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STRENGTH

OF

WROUGHT-IRON BRIDGE MEMBERS.

PART I.—GENERAL THEORY OF BEAMS.

PART II.—PRACTICAL FORMULAS FOR BEAMS, STRUTS, COLUMNS AND SEMI-COLUMNS.—EXTENDED COMPARISON OF VARIOUS FORMULAS WITH EXPERIMENT.

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PREFACE.

THE examination of existing bridges for strength and trustworthiness has occasioned the use of formulas not accessible to the writer in published form. In procuring the formulas by direct solution, the amount of labor entailed has been great. The avoidance of a repetition of this labor, on the part of others who may require these formulas, is believed to be sufficient reason for now publishing the results obtained.

The formulas referred to are such as take account of longitudinal as well as simultaneously acting transverse loads, while the beam is itself held in various ways.

In general formulas of the kind just referred to, it seems evident that either form of load may be made inferior, or to vanish altogether, giving formulas for ordinary beams, for columns, or for semi-columns, &c., as the case may be.

Thus, expressions forming the basis of rational formulas for columns have been obtained—formulas which, besides transcending the ordinary empirical formulas in coincidence of computed and experimental results, have, by making known the composition of constants, furnished a true key to the law of safe loads.

Also the effect of pin friction in pin bearings has been provided for, as well as the limited lengths between failure by crushing and by simple flexure.

The investigations leading to these results were instanced by the State Railway Inspection Service, under the Hon.

H. Sabine, Commissioner of Railroads and Telegraphs for Ohio, in which service results were sought, in critical cases, that were worthy of the utmost possible confidence. S. W. R.

OHIO STATE UNIVERSITY,
April 24, 1882.

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Strength of Wrought-Iron Bridge Members.

I.—General Theory of Beams.

Examples of beams coming under a more general theory than that of mere transverse loading are found in bridges; the same, for complete solution, requiring unusual formulas.

To make clear the nature of these formulas, the conditions to which certain bridge members are subject may be referred to. Thus, in some instances, the chords of truss bridges are required to carry the floor beams, one, two or more of them, to each panel. Now, when a train of cars comes upon a bridge, the floor-beam loads rest down upon the chord members and deflect them into downward bowing curves, causing "transverse strains," or "bending moments." Simultaneously, the load upon the bridge causes endlong strains in the same chord members; tension for the lower, and compression for the upper

chord. Thus, an individual member of the chord of the bridge, such as an "eye bar," is to be treated as a beam subjected to a combined bending and stretch, or bending and compression, as the case may be.

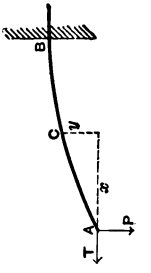
The usual way of calculating the resulting maximum strain in the piece considered, is to compute as though the beam had only the "cross strain," and then compute separately for the endlong strain, and add the results. It is evident, however, from a casual consideration that as the bending load would separately give a certain curve to the beam, the tension would partially straighten that curve and diminish the bending moment. Again, in a compressive endlong strain, the curvature of the beam would be increased and the bending moment increased. Hence, the usual calculation would give too large a value to the maximum strain per square inch in one case, and too small in the other. The formulas given in this article correct these anomalies.

The conditions of the piece just referred to as regards loading and supports are stated in general terms, but it is evident that they include all such special conditions as concentrated loads; distributed loads; one end fixed and the other free; one end fixed and the other supported; both ends supported; both ends fixed, &c.

TRUE FLEXURAL MOMENT AND ITS RESISTANCE IN MIXED STRAINS.

Before proceeding to special cases, it is necessary to consider the pseudo-philosophy of eccentric neutral axis in beams subject to both endlong and cross forces.

For the sake of fixing the ideas, let Fig. 1 represent a beam fixed at the end, B, and free at the end, A. Let the cross force, P, act at right angles to the beam, and the endlong force, T, in the direction of the length, both being applied at the free end of the beam. The bending will be as shown, the fibers on the upper side of the beam being stretched, and those on the lower side being compressed, if T is not too great.



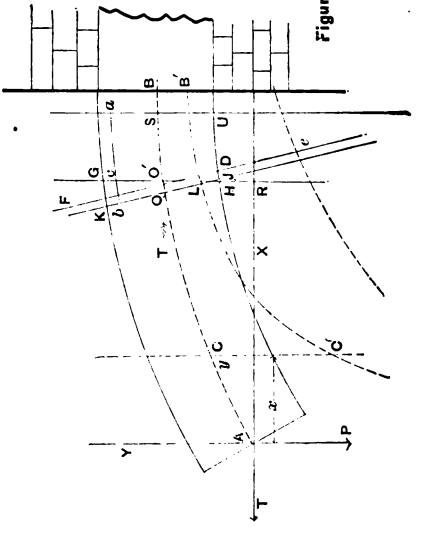


Figure I

Now, undoubtedly one effect of T is to elongate the whole beam and every part of it, throwing the neutral axis for all the stresses, downward from the line ACB to some line C′B′, or C″B″ This displacement is proportional to $\frac{T}{Px}$ for the section at the distance x, from the free end of the beam. If $T=o$, the displacement$=o$, anywhere. If $P=o$, the displacement is ∞, anywhere. If $x=o$, the displacement $=\infty$. If $x=l$, and P very small, the displacement is very great. But for T and P moderate, the displacement is moderate. For T and P constant and x variable, the departure of the line of the neutral axis, C′B′ from CB, increases as x decreases, and, geometrically speaking, it is asymptotic to the horizontal through B, and to the vertical, through A. Though this line is a real *neutral* axis, as regards stress, it can easily be shown that it is not the most convenient line in which to place the origin of moments, for the well-known differential equation of the elastic curve, viz.:

$$\varepsilon I \frac{d^2 y}{dx^2} = \Sigma P x = \Sigma(\text{applied moments}),$$
$$= M \quad . \quad . \quad (1)$$

This equation, as it stands, is an equation of moments of forces, the second member being the sum of the moments of the forces acting on the beam, while the 1st member is the sum of the moments of the internal forces of the beam resisting bending, or, more briefly, the "moment of resistance" of the beam. The origin of moments, or moment axis for this equation, is taken on some certain cross section of the beam, as, for instance, DF, and then the moments of the forces are to be expressed for that particular origin of moments. The equation is then integrated, resulting in the equation of the elastic curve. The effect of the integration is to move the plane, D F, from end to end of beam, thus summing the bending effects due to each element of length of beam.

That the origin of moments for the above equation of moments should be taken on the sectional plane, DF consid-

ered, no one doubts; but *where* it should be taken is seriously questioned. Some writers definitely locate it, by saying that the moment of inertia, I, is here variable, because the neutral axis, which is supposed coincident with the axis for I, is not parallel to the beam. According to this, if C'B' be the neutral axis, as above described, the origin of moments must be taken at L, in applying the above general equation (1) to Fig. 1. The moment of inertia, I, of the section, DF, is then to be expressed for the moment of inertia axis at L, coinciding with the origin of moments. Now, to discuss this conclusion, draw G H, DF, and JK, as shown. Let the portion, SO, of the beam be supposed to stretch, displacing DF to JK, this stretch being due to T. Again, let the bending moments displace GH, through an angle to FD, causing a stretch in the upper fibers and a compression in the lower fibers of the beam, the triangles, O'FG and O'HD, representing the two actions respectively. Respecting a fiber,

ab, the stretch from the line, G, to F is due to cross strain, and from F to K, to endlong strain.

Now, for the sake of an argument, let it be granted that the origin of moments for equation (1) be taken on the real neutral axis, C′B′, at L. This point is where the cross section, JK considered, intersects the real neutral axis. That is, L is the point in the section, J K, where there is actually no longitudinal stress. The figure indicates this, since at the top of the beam, GF stands for stretch due to cross strain; FK, the stretch for endlong strain, their sum, GK, being the total stretch or stress. This stress is less and less in going down from the top, until it vanishes at L, where the limiting lines, GL and KL, for the stress named, intersect on the neutral axis, C′B′.

Now, in applying equation (1) to the beam, Fig. 1, with the understanding that the *origin of moments be taken at* L, we have

$$\frac{\varepsilon I}{\rho} = \varepsilon I \frac{d^2 y}{d x^2} = \text{P.AR--T RL} \quad . \quad . \quad (2).$$

where the moment of inertia, I, of the first member, is to be so expressed that its axis coincides in position with the origin of moments, L. Let the beam be regarded as uniform in cross section from end to end, with the line, ACOB traversing the center of gravity of any cross section. Let I_1 represent the moment of inertia of the cross section considered, for an axis located at the center of gravity of that section. For KJ, this axis is at O, the point O being the "principal axis" or "center of inertia." Now, a well-known principle in mechanics, by which I may be obtained from I_1, gives

$$I = I_1 + \text{section JK}.\overline{LO}^2 \quad . \quad (3).$$

Hence the first member of equation (2) is

$$\frac{\varepsilon I}{\rho} = \frac{\varepsilon I_1}{\rho} + \frac{\varepsilon}{\rho} \text{ section JK}.\overline{LO}^2$$

$$= \frac{\varepsilon I_1}{\rho} + \frac{\varepsilon}{\rho} K.\overline{LO}^2 \quad . \quad . \quad (4).$$

where K is here taken for the cross section JK.

The denominator ρ is the radius of

curvature of the curve of the neutral axis as shown in Fig 1 by lines running down past *e*.

If S U and G H be taken parallel before flexure and fixed to the beam, SO being regarded as an element of length, dx, of the beam, then during flexure S U and G H intersect at the center of curvature of the elastic curve, thus determining ρ. Then, relative to the fiber S O′ O, Fig. 1, we will have the stretch of the fiber = O′ O ; and from the similarity of triangles we may write

$$\text{OL} : \rho : : \text{O}'\text{O} : \text{SO}'.$$

$$\therefore \quad \frac{\text{LO}}{\rho} = \frac{\text{O}'\text{O}}{\text{SO}'}.$$

A well-known fundamental formula in the theory of the elastic resistance of beams is

$$\text{T}\, l = \varepsilon\, \text{K}\, d\, l \quad . \quad . \quad . \quad (4\tfrac{1}{2}).$$

where T is a force applied to a bar of length, l, of section K, and co-efficient of elasticity, ε ; for which the stretch is $d\, l$. This formula may be applied to the part S O′ of the beam Fig. 1 by making $l =$

S O′ and $d\,l$=O′ O, the section, K, being the same whole section, JK. In this, the dl is=O′ O=F K=D J.

Hence

$$T.SO' = \varepsilon K.O'O = \varepsilon K.\frac{LO}{\rho}.SO'$$

or

$$T = \varepsilon K\frac{LO}{\rho}$$

combining this with (4), we get

$$\frac{\varepsilon I}{\rho} = \frac{\varepsilon I_1}{\rho} + T.LO$$

Comparing with equation (2) we have

$$\frac{\varepsilon I}{\rho} = \frac{\varepsilon I_1}{\rho} + T.LO = P.AR - T.RL$$

or

$$\frac{\varepsilon I_1}{\rho} = P.AR - T.RL - T.LO$$

$$= P.AR - T\,(RL + LO)$$

or

$$\frac{\varepsilon I_1}{\rho} = \varepsilon I_1\frac{d^2y}{dx^2} = P.AR - T.RO. = M$$

$$= \Sigma Px - \Sigma Ty \quad . \quad . \quad (5).$$

This equation has for its first member the ordinary expression for the moment of elastic resistance of beams, subject

simply to bending forces applied only at right angles to the beam; that is to say, the expression in which the origin of moments for the resisting forces is at the center of gravity of the cross section of beam considered, or at O, Fig. 1. The second member expresses simply the moments of the applied forces for the same origin of moments, viz.: the point O.*

Hence it follows that in cases of flexure of beams, where part of the forces are applied in an endlong or longitudinal direction, the moments may unhesitatingly be expressed with reference to an origin of moments located at the center of gravity of the cross section considered. Equation (5) makes this evident, not only for the moment of resistance but for the entire number of the applied

* In the above transformation the radius of curvature ρ has been all the time regarded as of the same value. Fig. 1 shows that this is correct. For in $\frac{\varepsilon I}{\rho}$ it extends from L to the center of curvature, or to where L J produced meets SU produced. In $\frac{\varepsilon I_1}{\rho}$ it extends from O′ to where O′D intersects SU produced. But as LJ and O′D are parallel to each other, the radii of curvature have the same value.

forces. Indeed, a little consideration of M, in equation (5), shows that the above statement respecting the moment of the applied forces is true in general; that is to say, it is true for any number of forces, having any conceivable points of application, and having any conceivable lines of direction. For each force may be resolved into two components, one longitudinal and the other transverse such as T.RO and P.AR respectively and added, ΣTy being the algebraic sum of the moments of all the former, and ΣPx of the latter. It is to be understood that in the integrations, proper limits are to be chosen, so that no point of concentration of applied force be overstepped in this, any more than in ordinary problems.

These facts place all problems in the resistance of beams on the same basis, that is to say, equation (5) applies with equal truth and exactness for a beam subject to the action of any system of forces, however complex, as for the simple case of a single transverse force; I_1

being always regarded as the principal moment of inertia and constant for a beam of uniform cross section.

This complete generalization of equations (1) and (5) for a constant value of I is important, since the integrations are not only facilitated, but we are enabled to apply the resulting formulas for a beam under any system of forces, with the same degree of confidence as for the most simple case.

To confirm the facts above stated by discussion, let us refer to Fig. 1. First, suppose P applied, bending the beam downward as shown. Then apply T as shown. This latter will evidently partially straighten the curvature, and raise the end of the beam. But it will finally come to rest under some degree of curvature, P and T both acting. The portion S O′ of the beam will stretch, D F moving to J K. Next apply an equal and opposite endlong force—T, at the center of gravity O, of the cross-section J K. This latter will have no tendency whatever to bend the portion S O, but it will

exactly return J K to D F. In this the end of the beam will not be raised or lowered, the deflection being entirely due to the two forces acting at the end of the beam. The stress in the beam is now simply that represented by the triangles O′ G F and O′ D H. The moment of resistance is $\frac{\varepsilon I_1}{\rho}$ axis for I_1 being at O′, and the moment of P will be P.AR. But as to T we have a *couple* between T at the line AR and—T at O′ the arm of the couple being RO. Hence the moment of this couple is T.RO′. All these moments are seen to have the origin of moments at the point O, or O′ both being practically the same point. This re-establishes equation (5).

So far, the endlong force, when spoken of in connection with Fig. 1, has been treated as producing tension. But it is easily seen that similar reasoning follows for T, so taken as to compress the beam. Indeed, the algebraic sum, ΣTy, mentioned above, contemplated positive as well as negative forces.

Equation (5) enables us to find the equation of the elastic curve of the bent beam. Having the equation of the curve, we can readily find the deflection at any point. Also the moment of strain M, may be found as soon as the unknown constants of integration are determined.

TRUE MAXIMUM STRAIN IN ANY CROSS SECTION.

By aid of the moment M and the end-long force T, we may find the maximum strain produced in any cross section of the beam, as for instance in the section at O, Fig. 1. At the top of the beam, the strain is GK, and at the bottom it is JH; GK being the greater, it is the one sought.

The strain GK is the sum of two parts, GF and FK, the former being due to the moment M of bending, and the latter to the direct tension T. To find the total amount of this strain per square inch, we may apply the well known formula for the strength of beams for finding the part GF thus

$$\frac{t_1 I_1}{d_1} = M, \quad . \quad . \quad . \quad . \quad (6).$$

where M is given by equation (5), as the algebraic sum of the moments of all the applied forces, the center of gravity of the section considered being the origin of moments, I_1 the *principal* moment of inertia for the same section, as previously stated, d_1 the distance from the center of gravity of the section to that point where the strain is greatest, or to where rupture or failure begins, as the case may be, and t_1 is the strain due to the moment M. In Fig. 1 O′ is the center of gravity of the section DF considered, I_1 is the moment o inertia of that section for O′ the axis, and d_1 is O′F. As a formula for determining the moment of ultimate resistance of beams, t_1 is to be assumed equal the ultimate resistance or modulus of resistance to rupture. Equation (6) is the converse of this, where M is predetermined and t_1 found as a stress considerably below the ultimate value. In calculating the ultimate strength of

beams, formula (6) is known to give values wide of the truth. This is due to the fact that the law of resistance in the neighborhood of rupture deviates greatly from the law of perfect elasticity. But within the elastic limits, as will usually be the case with (6), the formula, as is well known, will be truthful. The strain FK Fig. 1, as above explained is due to T, and if the intensity be represented by t_2 and the sectional area DF by K, we will evidently have

$$t_2 K = T \quad . \quad . \quad . \quad . \quad (7)$$

Let $t_1 = GK$, Fig. 1, stand for the whole maximum strain per square inch in the section considered.

Then

$$t = t_1 + t_2 = \frac{M d_1}{I_1} + \frac{T}{K} \quad . \quad . \quad . \quad (8).$$

a perfectly general formula, free from all assumed approximations except in the instance where ρ is placed equal the second differential co-efficient. This, however, is done by all writers on beams. But it is known that the error in this

assumption is of no practical import whatever in beams as ordinarily used. It is zero where the bent beam is not inclined to the axis x, and for an inclina- of 10° it is less than 5 per cent.

It is to be observed that for rupturing strains d_1 is to be taken as the distance from the center of gravity of the section to the point where the rupture takes place, whether it be by tension or compression.

When the beam is rectangular in cross section with a breadth b and depth d, we have

$$d_1=\frac{d}{2},\ \mathrm{I}_1=\frac{bd^3}{12},\ \text{and}\ \mathrm{K}=bd,$$

and hence (8) becomes,

$$t=\frac{6\mathrm{M}}{bd^2}+\frac{\mathrm{T}}{bd} \quad . \quad . \quad . \quad (9).$$

Here d_1 disappears, as indeed it will in all cases where the cross section of the beam is symmetrical with respect to an axis to that section taken horizontal through O′ Fig. 1.

NEUTRALIZING OF MOMENTS DUE TO CROSS STRAINS.

It is often desirable to so assume the point of application of T, as to place the beam under the least possible maximum stress when the structure containing the the beam receives its maximum load. For instance, suppose the lower chord of a bridge carries the floor beams, several to each panel. Then it is evident that T may be applied at such point as to make its moment, with respect to the section of beam in question, equal and opposite to the moment of P. This makes $M = 0$, and consequently $t_1 = 0$. This is evidently the best condition, since t_2 and T are sole and mutual dependencies, and, at best, the minimum strain can never fall below t_2. This, and also equation (8) agree in making $t_1 = 0$ and hence, for minimum strains,

$$M = o, \text{ and } t_1 = o.$$

To illustrate further, take an "eye bar" of a lower chord of any bridge. Let ABCD represent the loads due to

floor beams. The section at the middle of the length will receive the greatest moment of strain due to ABC, &c., while the tension T is uniform throughout. Now, that $t_1 = o$, the centers of the eyes must be raised a distance *ab*, shown at the middle section, so that for the maximum bridge load,

$$T.ab = \left\{ \begin{array}{l} \text{sum of the moments of AB and} \\ \text{P, with respect to } a \end{array} \right. \quad . \; . \; (10).$$

When this condition is exactly satisfied, the middle part of the beam near JK will be simply stretched and not sprung into curvature. But if the body of the bar be straight when free from strain, it will, when under the condition, (10) be sprung at points near the eyes. The problem then becomes complex because *ab* varies with the spring, and would require to be determined with due regard to the amount of flexure, as subsequently done in this article.

But it is possible to so curve the bar in the manufacture, that any part of it will be simply stretched when bearing its

the origin of co-ordinates being at the middle of one eye, with l the length of the bar, and w the loading per unit length.

In cases where the cross section of the beam is non-symmetrical, as often the case in upper chords of bridges, the center of gravity of that section is to be found and then ab laid off, downward in the upper chord and upward in the lower chord.

When chord members are very long, the moment due to their own weight should be found and ab determined by (11).

GENERAL INTEGRALS FOR SOLVING CASES OF MIXED STRAINS.

In the designing of new structures the precautions pointed out above relative to maximum stresses in mixed strains may be provided for in the simplest way, viz: by computing separately and adding, though this may not always be preferable. When this approximate method is not desirable, and in cases where existing

structures are to be examined, it may become necessary to make a careful determination of the moment M. To meet this requirement the various cases following have been worked out.

Before taking up the special cases an examination of the expressions for the moments of applied forces, shows that the integrations all come under two general forms, and it will be seen to be decidedly preferable to perform the two integrations once for all than to integrate for each special case.

The first general form is

$$\frac{d^2y}{dx^2} = Ay + Bx + Dx^2 + F \quad . \quad (12).$$

Differentiate twice, and

$$\frac{d^4y}{dx^4} = A\frac{d^2y}{dx^2} + 2D.$$

Now put

$$\frac{d^2y}{dx^2} = n,$$

and differentiate twice,

$$\frac{d^4y}{dx^4} = \frac{d^2n}{dx^2} = A\frac{d^2y}{dx^2} + 2D$$

or

$$\frac{d^2n}{dx^2}=An+2D$$

multiply through by dn, and integrate, and we get

$$\frac{dn^2}{dx^2}=An^2+4Dn+C$$

or

$$dx=\frac{dn}{\sqrt{A}\sqrt{n^2+\frac{4Dn}{A}+\frac{C}{A}}}$$

whence, by integration

$$x=\frac{1}{\sqrt{A}}\text{hyp. log}\frac{\frac{2D}{A}+n+\sqrt{n^2+\frac{4Dn}{A}+\frac{C}{A}}}{C_1}$$

Passing to exponentials and transposing in part,

$$C_1e^{x\sqrt{A}}-\frac{2D}{A}-n=\sqrt{n^2+\frac{4Dn}{A}+\frac{C}{A}}$$

By squaring both members and canceling

$$C_1^2e^{2x\sqrt{A}}-\frac{4D}{A}C_1e^{x\sqrt{A}}-2C_1ne^{x\sqrt{A}}+\frac{4D^2}{A^2}=\frac{C}{A}$$

solving for n we obtain

$$\left.\begin{aligned} n=&\frac{C_1}{2}e^{x\sqrt{A}}-\frac{C}{2C_1A}e^{-x\sqrt{A}} \\ &-\frac{2D}{A}+\frac{2D^2}{C_1A^2}e^{-x\sqrt{A}} \\ =&\frac{d^2y}{dx^2}=Ay+Bx+Dx^2+F \end{aligned}\right\} (13)$$

The second general form is

$$\frac{d^2y}{dx_2}=-Ay-Bx-Dx^2-F \quad . . \quad (14).$$

Differentiate twice and

$$\frac{d^4y}{dx^4}=-A\frac{d^2y}{dx^2}-2D$$

Put

$$\frac{d^2y}{dx^2}=+n$$

and differentiate twice

$$\frac{d^4y}{dx^4}=\frac{d^2n}{dx^2}=-A\frac{d^2y}{dx^2}-2D$$

or

$$\frac{d^2n}{dx^2}=-An-2D$$

Multiply through by dn and integrate and

$$\frac{dn^2}{dx^2} = -An^2 - 4Dn + C'$$

or

$$dx = \frac{dn}{\sqrt{C' - 4Dn - An^2}}$$

But

$$\int \frac{dz}{\sqrt{a \pm bz - cz^2}} = \frac{1}{\sqrt{c}} \sin^{-1} \frac{2cz \mp b}{\sqrt{4ac + b^2}}$$

where $a = C'$, $b = 4D$, $c = A$, $z = n$.

Hence

$$x = \frac{1}{\sqrt{A}} \sin^{-1} \frac{2An + 4D}{\sqrt{4AC' + 16D^2}} - C',$$

or

$$\sin(x\sqrt{A} + C_1') = \frac{An + 2D}{\sqrt{AC' + 4D^2}}$$

Solving for n, we obtain

$$\left.\begin{aligned} n &= \frac{\sqrt{AC' + 4D^2}}{A} \sin(x\sqrt{A} + C_1') \\ -\frac{2D}{A} &= \frac{d^2y}{dx^2} = -Ay - Bx - Dx^2 - F \end{aligned}\right\} (15).$$

In the application of (13) and (15) to the following cases, it is only necessary to cut out any redundant term, and determine the constants of integration.

SOME OF THE PRINCIPAL CASES UNDER MIXED STRAINS.

I. *For the first case let the beam be fixed at one end, free at the other, and having a load* P *and pull* T *applied at the free end.*

Fig. 1 or 1a represents it. The first question arising is whether to apply (13) or (15). An inquiry will reveal the fact that the sign of Ty determines which. The calculus demands that the

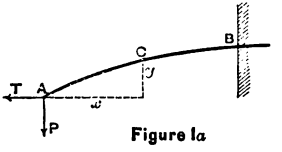


Figure 1a

ordinate and second differential coefficient have contrary signs when the curve is concave toward the axis of x. Hence in Fig. 1 they are contrary. Taking T positive when it is directed toward the beam, and negative when from it, we will have, for T pointing outward as in Fig 1 or 1a.

$$-\varepsilon I_1 \frac{d^2y}{dx^2} = Px - Ty = M$$

T being negative makes the sign of $\frac{d^2y}{dx^2}$ and Ty alike.

For convenience let the subscript to I be dropped in the following equations, I being always regarded as the principal moment of inertia of the cross section of the beam. Then we have for the origin of moments at C, Fig. 1a, and of coordinates at A,

$$\varepsilon I \frac{d^2y}{dx^2} = Ty - Px = M \quad \ldots \quad (16).$$

To integrate this equation, we require (13) because the signs of Ty and Ay are alike, also of the second differential coefficients.

Comparing with (13)

$$A = \frac{T}{\varepsilon I} \quad B = -\frac{P}{\varepsilon I} \quad D = o \quad F = o$$

Hence

$$\frac{d^2y}{dx^2} = Ay + Bx = \frac{C_1}{2} e^{nx} - \frac{C}{2C_1 A} e^{-nx}$$

where for convenience $n = \sqrt{A} = \sqrt{\frac{T}{\varepsilon I}}$

n here differing from the n used in the integrations for (13) and (15).

The above equation for this case is given in *Strength of Materials*, by De Volson Wood, p. 300, 2d ed.

Let the length of the beam be l, and the distance of the point of application of T below the center of gravity of the end of the beam be h. Then if $x=o$ $y=h$, and

$$Ah=\frac{C_1}{2}-\frac{C}{2C_1A}$$

$\therefore$

$$\frac{C}{2C_1A}=\frac{C_1}{2}-Ah$$

and

$$Ay+Bx=\frac{C_1}{2}\left(e^{nx}-e^{-nx}\right)+Ahe^{-nx}$$

$$A\frac{dy}{dx}+B=\frac{C_1}{2}\left(e^{nx}+e^{-nx}\right)-Ahne^{-nx}$$

But

$$\frac{dy}{dx}=o \text{ for } x=l \text{ and}$$

$$B=\frac{C_1}{2}n\left(e^{nl}+e^{-nl}\right)-Ahne^{-nl}$$

The value of C_1 from this, introduced above, gives

$$Ay+Bx=\frac{\frac{B}{n}+Ahe^{-nl}}{e^{nl}+e^{-nl}}\left(e^{nx}-e^{-nx}\right)+Ahe^{-nx}$$

Restoring the values of A and B, we get

$$Ty-Px=-\left(\frac{P}{n}-The^{-nl}\right)\frac{e^{nx}-e^{-nx}}{e^{nl}+e^{-nl}}+The^{-nx}$$

$$=M \quad . \quad . \quad . \quad . \quad . \quad (17).$$

This equation expresses the relation between x and y for the equation of the curve of the axis of the beam, that is to say, of the curve AOB, Fig. 1 or 1a. The maximum deflection of the beam is y_1-h for $x=l$.

Also (17) is so arranged as to give the moment of the applied forces for any point, the first member being the same as (16).

If $x=o$ $y=h$ as it should.

The maximum moment of strain for the whole beam is evidently at the fixed end where $x=l$ and $y=y_1$. Hence, by reduction,

$$Ty_1 - Pl = -\frac{\frac{P}{n}\left(e^{nl} - e^{-nl}\right) - 2Th}{e^{nl} + e^{-nl}}$$

$$= M_{max}. \quad . \quad . \quad . \quad (18).$$

When $h = o$,

$$Ty_1 - Pl = -\frac{P}{n}\left\{\frac{e^{nl} - e^{-nl}}{e^{nl} + e^{-nl}}\right\} = M_{max}. \quad (19).$$

with h restored and $P = o$

$$Ty_1 = \frac{2Th}{e^{nl} + e^{-nl}} \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Evidently the resultant of T and P might be substituted, and be considered as a single force acting obliquely. Conversely any single force acting obliquely at the end of the beam could be resolved into components corresponding with T and P.

If we imagine this resultant acting at the end of h, its prolongation would intersect the beam when h is downward. This point of intersection would evidently be a point of contrary flexure at which the moment would be zero. The

point could be found by placing the 2d member of (17) $= o$ and solving for x. This point would be a neutral point for an S shaped curve,

II. *Let the conditions be as in* I, *except change* T *to a push.*

This would seem to be met by changing the sign of T, but this makes all the expressions in case I imaginary. But it may be rationalized as done in Prof.

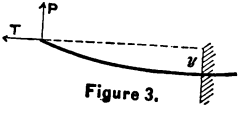


Figure 3.

Wood's *Strength of Materials*, p. 301, 2d ed. But if we change the sign of P instead of T, the relation of signs of T and I is right. In applying equation (19) with sign of P changed, we will obtain a negative result for y, while when P is positive y is positive. Hence the conditions for this supposition are as shown in Fig. 3, with no essential change. It is simply case I turned over. The moment

obtained from (19) is the same numerically for P + as for − which is evidently as it should be, while for T changed in sign it would be greater.

Hence, changing the sign of T, our equation of moments for this case becomes, as evident from Fig. 4, for the

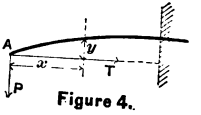


Figure 4.

origin of moments at C and of co-ordinates at A.

$$\varepsilon I\frac{d^2y}{dx^2}=-Ty-Px=M \quad \ldots \quad (21)$$

This may be integrated by aid of equation (15), as the signs are proper to it, where,

$$A=\frac{T}{\varepsilon I}=n^2 \quad B=\frac{P}{\varepsilon I} \quad D=o \quad F=o$$

Hence the integral is

$$-Ay-Bx=\sqrt{\frac{C}{A}}\sin(x\sqrt{A}+C_1)$$

$$=\sqrt{\frac{C}{A}}\sin\ (nx+C_1)$$

For $x=o$ let $y=h$ and

$$-\mathrm{A}h=\sqrt{\frac{\mathrm{C}}{\mathrm{A}}}\sin\,\mathrm{C}_1$$

$$-\mathrm{A}y-\mathrm{B}x=-\frac{\mathrm{A}h}{\sin\,\mathrm{C}_1}\sin(nx+\mathrm{C}_1)$$

Differentiate and,

$$-\mathrm{A}\frac{dy}{dx}-\mathrm{B}=-\frac{\mathrm{A}hn}{\sin\,\mathrm{C}_1}\cos(nx+\mathrm{C}_1)$$

But

$$\frac{dy}{dx}=o \text{ for } x=l$$

$$\therefore \quad \mathrm{B}=\mathrm{A}hn(\cos\,nl\,\cot\mathrm{C}_1-\sin\,nl)$$

or

$$\mathrm{Cot}\,\mathrm{C}_1=\frac{\mathrm{B}}{\mathrm{A}hn\,\cos\,nl}+\mathrm{tang}\,nl$$

Hence

$$-\mathrm{A}y-\mathrm{B}x=-\frac{\mathrm{B}}{n}\frac{\sin\,nx}{\cos\,nl}$$
$$-\mathrm{A}h\,(1+\mathrm{tang}\,nx\,\mathrm{tang}\,nl)\,\cos\,nx$$

or, restoring the values of A, B and n, we get the general equation of the curve of the axis of the beam, also the moment at any point,

$$-\mathrm{T}y-\mathrm{P}x=-\frac{\mathrm{P}\sin nx}{n\cos nl}-$$
$$\mathrm{T}h(1+\mathrm{tang}nx\mathrm{tang}nl)\cos nx=\mathrm{M}\ .\ (22).$$

Here also we will find a point of contrary flexure, and of zero moment of flexure, where the resultant of P and T may, when prolonged, intersect the beam.

The maximum moment of flexure occurs where $x=l$ and $y=y_1$, which is

$$-Ty_1-Pl=-\frac{P}{n}\text{tang}nl-\frac{Th}{\cos nl}=M._{max} \quad . \quad . \quad (23).$$

when $h=o$

$$-Ty_1-Pl=-\frac{P}{n}\text{tang}nl=M._{max} \quad . \quad (24).$$

When h is restored, and $P=o$

$$-Ty_1=-\frac{Th}{\cos nl}=M \quad . \quad . \quad (25).$$

III. *Let the beam be conditioned as represented in Fig. 5, viz.: fixed at one*

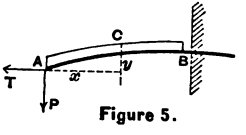


Figure 5.

end, free at the other, with a load P *and a pull* T *at the free end, and a uniform load over its length.*

Let w = distributed load per unit length of beam.

Then the equation of moments will be

$$\varepsilon I\frac{d^2y}{dx^2}=Ty-Px-\frac{wx^2}{2} \quad . \quad . \quad . \quad (26).$$

Hence

$$+Ay+Bx+Dx^2=\frac{C_1}{2}e^{nx}-$$

$$\frac{Ce^{-nx}}{2C_1n^2}-\frac{2D}{n^2}+\frac{2D^2}{C_1n^4}e^{-nx}$$

where

$$A=\frac{T}{\varepsilon I}=n^2 \quad B=-\frac{P}{\varepsilon I} \quad D=-\frac{w}{2\varepsilon I}$$

To determine the constants of integration make x and y zero for the point A. This gives

$$\frac{C}{2C_1n^2}=\frac{C_1}{2}-\frac{2D}{n^2}+\frac{2D^2}{C_1n^4}$$

Introducing this, differentiating, and making $\frac{dy}{dx}=o$ for $x=l$ we get

$$\frac{C_1}{2}=\frac{B+2Dl+\frac{2D}{n}e^{-nl}}{\left(e^{nl}+e^{-nl}\right)n}$$

which introduced, gives us the following general equation of moments, and of the curve of the axis of the beam, T P and w being restored,

$$Ty - Px - \frac{wx^2}{2} = -$$

$$\frac{P + wl + \frac{w}{n}e^{-nl}}{n}\left\{\frac{e^{nx} - e^{-nx}}{e^{nl} + e^{-nl}}\right\} + \frac{w}{n^2}\left(1 - e^{-nx}\right) \quad . \quad . \quad . \quad . \quad (27).$$

In this equation the sign of P may be + or −; that is, P may act down or up. Under certain circumstances the beam will be S shaped, with a point of contra flexure, or of zero moment.

To ascertain this point place the second member $= o$ and solve for x.

The maximum moment is evidently obtained by making $x = l$. The equation then will reduce to

$$Ty_1 - Pl - \frac{wl^2}{2} = -\frac{P + wl}{n}\left\{\frac{e^{nl} - e^{-nl}}{e^{nl} + e^{-nl}}\right\} + \frac{w}{n^2}\left\{1 - \frac{2}{e^{nl} + e^{-nl}}\right\} = M_{max} \quad . \quad (28).$$

If $w=o$ this equation reduces to (19), as it evidently should. Also P may be made zero, leaving simply the uniform load and T.

IV. *Let the condition be as in Fig. 5, except reverse* T, *as in Fig. 6.*

Then

$$\varepsilon I\frac{d^2y}{dx^2}=-Ty-Px-\frac{wx^2}{2} \quad . \quad . \quad (29).$$

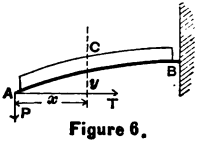


Figure 6.

Applying equation (15) for integrating this

$$-Ay-Bx-Dx^2=\frac{\sqrt{AC+4D^2}}{A}\sin(nx+C_1)-\frac{2D}{A}$$

where $A=\frac{T}{\varepsilon I}$ $B=\frac{P}{\varepsilon I}$ $D=\frac{w}{2\varepsilon I}$

To determine the constants of integration make $x=o$ for $y=o$, and eliminate the coefficient to 1st term, 2d member.

Then make the tangent tc the curve zero for $x=l$, and eliminate C_1, as was done in case II., we then get

$$-Ty-Px-\frac{wx^2}{2}=-\frac{P+wl}{n}\frac{\sin nx}{\cos nl}-\frac{w}{n^2}(1-\cos.nx-\sin.nx\,\text{tang}nl)=M \quad (30).$$

The load P may be + or —.

Making $x=l$ for the maximum moment, we get, by reduction,

$$-Ty_1-Pl-\frac{wl^2}{2}=-\left(\frac{P+wl}{n}-\frac{w}{n^2}\text{tang}\frac{nl}{2}\right)$$
$$\text{tang}\, nl=M_{max} \quad . \quad . \quad (31).$$

If $w=o$, this expression reduces to (24), as it obviously should. If $P=o$ we have simply the uniform load w, and the end thrust T.

V. *Let the beam be supported at both ends, carry a uniformly distributed load, and have a pull applied at both ends.*

The conditions are obviously equivalent in the two parts of Fig. 7, the latter being already solved in equations (27) and (28).

The equation of moments for this figure is

$$\varepsilon I\frac{d^2y}{dx^2}=Ty-\frac{wlx}{2}+\frac{wx^2}{2} \quad . \qquad (32).$$

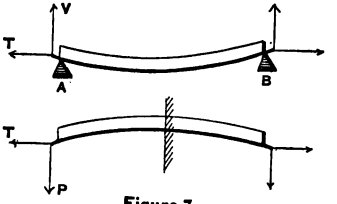


Figure 7.

since $V=\frac{wl}{2}$. Comparing with (27) we find the 1st member changed to (32) by changing P, $-w$ and l there, to $\frac{wl}{2}$, w and $\frac{l}{2}$ here. By making the corresponding changes in 2d member of (27) we obtain for the 2d member of (32),

$$+\frac{w}{n^2}\left\{\frac{e^{n\left(\frac{l}{2}-x\right)}+e^{-n\left(\frac{l}{2}-x\right)}}{e^{n\frac{l}{2}}+e^{-n\frac{l}{2}}}-1\right\} \qquad (32).$$

being l the whole length, AB.

For the maximum moment $x=\frac{l}{2}$ and

$$Ty_1-\frac{wl^2}{8}=\frac{w}{n^2}\left\{\frac{2}{e^{\frac{nl}{2}}+e^{-\frac{nl}{2}}}-1\right\}$$

$$M_{max.} \quad . \quad . \quad . \quad (33).$$

VI. *Reverse* T; *otherwise take the case as in* V.

Hence reverse w, and take $P=\frac{wl}{2}$ in equations (30) and (31) and change l to $\frac{l}{2}$ and we obtain

$$-Ty-\frac{wlx}{2}+\frac{wx^2}{2}=$$

$$\frac{w}{n^2}\left(1-\cos nx-\sin nx \operatorname{tang} n\frac{l}{2}\right)=M \quad (34).$$

l being the whole length AB, and for the maximum moment $x=\frac{l}{2}$. $\therefore$

$$\therefore \quad -Ty_1-\frac{wl^2}{8}=-\frac{w}{n^2}\operatorname{tang} n\frac{l}{4}\operatorname{tang} n\frac{l}{2}$$

$$M_{max} \quad . \quad . \quad . \quad (35).$$

VII. *Let the beam be supported at its ends, with a load* Q *placed at equal dis-*

tances from the ends, and have à tension T *applied.*

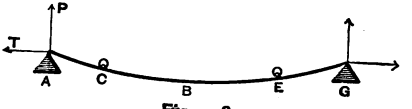


Figure 8.

1st. Then for the part AC, the moments will be

$$\varepsilon I\frac{d^2y}{dx^2}=Ty-Px=Ty-Qx \quad . \quad . \quad (36),$$

comparing with eq. (13) we get

$$Ay+Bx=\frac{C_1}{2}e^{nx}-\frac{C}{2C_1A}e^{-nx} \quad . \quad (13).$$

where

$$A=\frac{T}{\varepsilon I}=n^2 \quad B=-\frac{P}{\varepsilon I} \text{ and } P=Q \quad . \quad (37).$$

For $x=o$, $y=o$ and

$$\frac{C}{2C_1A}=\frac{C_1}{2}$$

$$Ay+Bx=\frac{C_1}{2}\left(e^{nx}-e^{-nx}\right)$$

Differentiate and let $\frac{dy}{dx}=\text{tang } i$ for $x=a$ $=AC=EG$, and

$$\frac{C_1}{2}=\frac{A\ \text{tang}\,i+B}{\sqrt{A}\left(e^{na}+e^{-na}\right)}$$

$$\therefore\quad Ay+Bx=\frac{A\ \text{tang}\,i+B}{\sqrt{A}}\left\{\frac{e^{nx}-e^{-nx}}{e^{na}+e^{-na}}\right\} \quad \ldots\ (39).$$

2d. For the part CB, the moments will be

$$\varepsilon I\frac{d^2y}{dx^2}=Ty+Q(x-a)-Px=Ty-Qa$$

where

$$A=\frac{T}{\varepsilon I}=n^2 \text{ and } F=-\frac{Qa}{\varepsilon I} \quad \ldots\ (40).$$

$$\therefore\quad Ay+F=\frac{C'}{2}e^{nx}-\frac{C_2}{2C'A}e^{-nx} \quad \ldots\ (13).$$

Now differentiate this and make $\frac{dy}{dx}=o$ for $x=\frac{l}{2}$ and

$$\frac{C_2}{2C'A}=-\frac{C'}{2}e^{nl}$$

Put this back in the same equation and make $\frac{dy}{dx}=\text{tang}\ i$ for $x=a$, and

$$A\ \text{tang}\,i=\frac{C'}{2}\left(e^{na}-e^{n(l-a)}\right)\sqrt{A}$$

This determines C′ and C_2, and the values placed in the integral above give

$$Ay+F=\frac{A \text{ tang } i}{\sqrt{A}}\left\{\frac{e^{nx}+e^{n(l-x)}}{e^{na}-e^{n(l-a)}}\right\} . \quad (41).$$

Now making the ordinates y, in the two equations, equal each other for the point C, where $x=a$ we obtain observing that $Ba=F$, (39)=(41), giving an expression by aid of which tang i is found. This value of tang i, put into eq. (39) or (41), gives the required expressions for the moments on the parts AC and BC, respectively.

Placing the parenthetical part of (39) equal L, and of (41) equal N, we get

$$A \text{ tang} i=-\frac{BL}{L-N} \quad . \quad . \quad (42)$$

The maximum moment on the whole beam will be found at the points C and E, directly under the equal loads Q, equidistant from the ends. This is obvious, from the fact that the moment due to P and Q for any part of CE is constant; but the moment of T, counter to that of

P and Q, is least at C and E, for the part CE. As to AC, the moment is greatest at C.

Hence (39) or (41) will give the max. moment which is at $x=a$

$$Ty_1-Qa=\frac{Q}{2n}\frac{e^{n\frac{l}{2}}-e^{-n\frac{l}{2}}+e^{-n\left(\frac{l}{2}-2a\right)}-e^{n\left(\frac{l}{2}-2a\right)}}{e^{n\frac{l}{2}}+e^{-n\frac{l}{2}}} \quad (43).$$

$$=M_{max.}$$

If we make $a=o$ the moment (43) reduces to zero, as it evidently should, as it leaves the beam without load. Again, if $a=\frac{l}{2}$, the expression reduces to the same form as (19), P and l there, being equivalent to Q and $\frac{l}{2}$ here, a fact that verifies all complex and extended reductions of this case.

VIII. *Let this beam be conditioned the same as in* VII., *except reverse* T, *as shown in Fig. 9.*

The expression for the moment on AC is

$$\varepsilon I\frac{d^2y}{dx_2}=-Ty-Px,$$

where

$$A=\frac{T}{\varepsilon I}=n^2 \qquad B=\frac{P}{\varepsilon I}=\frac{Q}{\varepsilon I} \quad . \quad (44).$$

Figure 9.

Hence 1st by (15), making $x=o$ for $y=o$, for one constant, and $\frac{dy}{dx}=\text{tang } i$ at $x=a$ $=AC=$ EG for the other constant.

$$-Ay-Bx=-(\sqrt{A}\text{tang}i+B)\frac{\sin nx}{\cos na} \quad (45).$$

the eq. of the axis AC, and of moments for the same.

2d. For the part CB, moment=

$$\varepsilon I\frac{d^2y}{dx^2}=-Ty-Px+Q(x-a)=-Ty-Qa$$

where

$$A=\frac{T}{\varepsilon I}=n, \quad F=\frac{Qa}{\varepsilon I}=\frac{Pa}{\varepsilon I}=Ba$$

Applying (15), first making $\frac{dy}{dx} = o$ fo

$x = \frac{l}{2}$ for one constant; then making th

same equation = tang i for $x = a$, for th

other, we get

$$-Ay - F = -\frac{\sin\left(nx + \frac{\pi}{2} - n\frac{l}{2}\right)}{\cos\left(na + \frac{\pi}{2} - n\frac{l}{2}\right)} \sqrt{A}\,\text{tang}$$

. . . (46

Then equating the ordinates y_1 from (4

and (46) for $x = a$, observing that $Ba = I$

we get the expression containing tang.

Substituting it in (46) we get the gener

expression of moments for the part CI

also the equation of the curve of the ax

of that part,

$$-Ty - Qa = -\frac{Q}{n}$$

$$\text{tang}na.\sin\left(nx + \frac{\pi}{2} - nl\right)$$

$$\sin\left\{\frac{\pi}{2} - n\left(\frac{l}{2} - a\right)\right\} - \tan na.\cos\left\{\frac{\pi}{2} - n\left(\frac{l}{2} - a\right)\right.$$

$$= M \quad . \quad . \quad . \quad (47$$

For $x=a$ this becomes

$$-\frac{Q}{n}\frac{\text{tang } na}{1-\text{tang}na\cot\left\{\frac{\pi}{2}-n\left(\frac{l}{2}-a\right)\right\}}$$
$$=M_a \quad . \quad . \quad . \quad (48).$$

For $x=a=\frac{l}{2}$ we get

$$-\frac{Q}{n}\text{tang}n\frac{l}{2}=M_{max} \quad . \quad . \quad (49).$$

(49) being greater than (48). This agrees with the indications of Fig. 9.

Equation (49) is the same as (24) as it evidently should.

If $a=o$ the moment reduces to zero, which the figure shows to be correct.

IX. *Let the beam be supported at the ends, be under tension* T, *and loaded with Q applied at any point.*

1st. For the part AC we get, origin of co-ordinates at A,

$$\varepsilon I\frac{d^2y}{dx^2}=Ty-Px$$

Applying (13), observing that for $x=o$ $y=o$ for one constant: and that $\frac{dy}{dx}=$

tang i, for $x=a$ for the other, and reducing we get

$$Ty-Px=\frac{T.\text{tang}\,i-P}{n}\left\{\frac{e^{nx}-e^{-nx}}{e^{na}+e^{-na}}\right\} \quad (50).$$

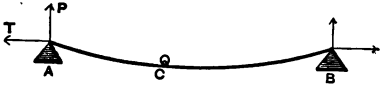


Figure 10.

From the figure $Pl=Q(l-a)$

$$\therefore \qquad P=Q\left(1-\frac{a}{l}\right) \quad . \quad . \quad . \quad (51).$$

If in (50) we make $a=\frac{l}{2}$ we have tang $i=o$, and introducing P, (50) reduces to the same as (19) and (43) for like values of a, and P.

2d. For the part BC with some reduction,

$$\varepsilon I\frac{d^2y}{dx^2}=Ty+\frac{Qa}{l}x-Qa$$

Applying (13) again, observing that for $x=l$ $y=o$, and $\frac{dy}{dx}=$tang i, for $x=a$, and reducing we get

$$\mathrm{T}y+\frac{\mathrm{Q}a}{l}x-\mathrm{Q}a=$$

$$\frac{\mathrm{T}.\mathrm{tang}\,i+\mathrm{Q}-\mathrm{P}}{n}\left\{\frac{e^{nx}-e^{n(2l-x)}}{e^{na}+e^{n(2l-a)}}\right\}\quad(52).$$

This equation reduces to the same as (50), (43) and (19) for like values of P, by making $a=\frac{l}{2}$; except (52) will have the negative sign, as it should, since its moment at C should oppose that of (50).

Now make $x=a$, and equate (50) and (52). This determines tang i. This can be restored to (50) and (52) for moments, or for equations of the curves of the axis of the parts of the beam. Thus the conditions of the beam are completely determined.

If we make the parenthetical part of (50) and (52)=L and N, respectively, we find, for $x=a$, in L and N,

$$\mathrm{tang}\,i=\frac{\mathrm{Q}}{n^2}\left(\frac{\mathrm{L}}{\mathrm{L}-\mathrm{N}}-\frac{a}{l}\right)\quad\ldots\quad(53).$$

For $a=\frac{l}{2}$; this reduces to o as it should.

In seeking the maximum moment we suspect that when Q is near A, or B at a particular point, the combined moment of T and P may possibly have a maximum between Q and the remote end, because that due to T varies as the ordinate y and P with x. But the mathematical test applied to (52) develops no such max. Hence the max. is always at the load Q, and can readily be found at $x=a$.

X. *Let the conditions be the same as in* IX. *except reverse* T, *or put the beam under compression, as in Fig. 11.*

Figure 11.

1st. For part AC

$$\varepsilon I\frac{d^2y}{dx^2}=-Ty-Px$$

where

$$P=Q\frac{l-a}{l}$$

Integrating by aid of (15), making x and $y=o$ for one constant, and $\frac{dy}{dx}=$ tang i for the other, we obtain

$$-\mathrm{T}y-\mathrm{P}x=-\frac{\mathrm{T}\,\mathrm{tang}i+\mathrm{P}}{n}\,\frac{\sin nx}{\cos na}\quad .\quad (54).$$

2d. For the part BC we have

$$\varepsilon\mathrm{I}\frac{d^2y}{dx^2}=-\mathrm{T}y-(\mathrm{P}-\mathrm{Q})x-\mathrm{Q}a$$

Applying (15) and make $x=l$ for $y=o$ for one constant, and $\frac{dy}{dx}=$tang i at $x=a$ for the other, and reducing we obtain

$$-\mathrm{T}y-(\mathrm{P}-\mathrm{Q})x-\mathrm{Q}a=$$

$$\frac{\mathrm{T}\,\mathrm{tang}i+\mathrm{P}-\mathrm{Q}}{n}\,\frac{\sin n(l-x)}{\cos n(l-a)}\quad .\quad (55).$$

Equating 2d members of (54) and (55) for tang i at $x=a$, and then replacing it; the same equations become, for AC,

$$-\mathrm{T}y-\mathrm{P}x=$$

$$\frac{\frac{\mathrm{Q}}{n}}{1+\mathrm{tang}na\cot n(l-a)}\cdot\frac{\sin nx}{\cos na}\quad .\quad (56),$$

and for BC,

$$-\mathrm{T}y-(\mathrm{P}-\mathrm{Q})x-\mathrm{Q}a=-\frac{\frac{\mathrm{Q}}{n}}{1+\operatorname{tang} na \cot n(l-a)}\cdot\frac{\sin n(l-x)}{\cos n(l-a)} \quad (57).$$

which are equations of the axial curves and of moments.

These reduce to the same as (49) for $x=a=\frac{l}{2}$, also to (24) for proper value of l. Thus (56) becomes

$$\frac{\mathrm{Q}}{2n}\operatorname{tang} n\frac{l}{2}=\mathrm{M}_{\max} \quad . \quad (58).$$

the same as (48), observing that Q in (48) equals 2Q in the above.

In looking for the max. moment, we observe that when Q is near A, the moment will without question increase from A to C, because the moment arms of both T and P increase together. Also the moment of the reaction of the support B will increase from B to C. But the moment of T is greatest where the curve of the axis of the beam is lowest, and this is between Q and B. Now when T is great and P small, particularly when T

is sufficient to hold the beam in the curved form, the maximum is plainly seen to be somewhat removed from Q towards the middle of beam, but in no case beyond the middle.

The 2d member of equation (57) is the expression for the moment of flexure of BC. Hence the moment varies along BC as the ordinates of a sinusoid the max. value being where sin. $n(l-x)=1$, or where $n(l-x')=\frac{\pi}{2}$. Hence

$$\therefore \quad \frac{x'}{l}=1-\frac{\pi}{2nl}$$

is the fractional part of the beam's length at which the moment *may* be a maximum, and where it will exist, in case BC has a less value. At $x=l$, or at B, the sinusoid starts, and overspans the beam length AB, as appears from the fact that when $x=o$, the value of (57) is still considerable, and positive. To find the springing points of the curve, place

$$\sin n(l-x)=o$$

giving

$$n(l-x)=o \text{ or} =\pi$$

Hence

$$x=l, \text{ or} =l-\frac{\pi}{n} \quad . \quad . \quad . \quad (59).$$

and the sinusoid is BED, for the beam AB, Fig. 12. The max. is at H when

$$AH=l-\frac{\pi}{2n} \quad . \quad . \quad . \quad (60).$$

equations (59) and (60) show that BD is twice BH. Suppose $n=.01$ and $l=200$. Then AH=43. and AD=−114.

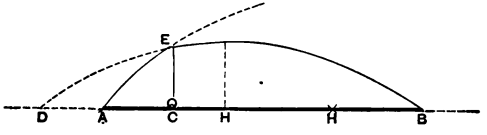


Figure 12.

The figure shows how the max. moment may occur at H when Q is at C, the curve AE being the sinusoid of eq. (56).

Similarly it may be shown that another point H′ exists where BH′=AH.

Now when Q is placed between H

and H′, the max.. moment is at Q, but it is at H or H′ when Q is outside those points.

XI. *Let the beam be fixed at one end, supported at the other, under compression T, and loaded with Q at any point.*

Take the origin of co-ordinates at A, y positive downward and $Ac=a$. Then

1st for the part AC (Fig. 13)

$$\varepsilon I\frac{d^2y}{dx^2}=-Ty-P(l-x)+Q(a-x)$$

$$=-Ty-(Q-P)x-(Pl-Qa)$$

Integrating by aid of (15) we get

$$-Ay-Bx-F=\frac{\sqrt{AC}}{A}\sin(nx-C_1)$$

where

$$A=\frac{T}{\varepsilon I}=n^2\quad B=\frac{Q-P}{\varepsilon I}\quad y=\frac{Pl-Qa}{\varepsilon I}$$

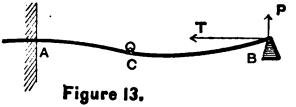


Figure 13.

For x and $y=o$ we get

$$\frac{\sqrt{C}}{n}=-\frac{F}{\sin C_1}$$

and for $\frac{dy}{dx}=o$ for $x=o$ we get

$$\cot C_1=\frac{B}{nF}$$

Hence

$$-Ay-Bx-F=-\frac{B}{n}\sin nx \\ -F\cos nx \quad . \quad (61).$$

and

$$-A \text{ tang } i-B=-B\cos nx+nF\sin nx$$

2d, For the part BC

$$\varepsilon I\frac{d^2y}{dx^2}=-Ty-P(l-x)$$
$$=-Ty+Px-Pl$$

Hence by (15)

$$-Ay-B'x-F'=\frac{\sqrt{C'}}{n}\sin(nx+C_1')$$

where

$$A=\frac{T}{\varepsilon I}=n^2,\ B'=-\frac{P}{\varepsilon I},\ F'=\frac{Pl}{\varepsilon I}.$$

also

$$B'l+F'=o, B-B'=\frac{Q}{\varepsilon I},\ a(B-B')=(F'-F)$$

For $x=l$, $y=o$ and

$$-B'l-F'=\frac{\sqrt{C'}}{n}\sin(nl+C_1')=o$$

$$\therefore \quad C_1'=-nl$$

$$\therefore -Ay-B'x-F'=\frac{\sqrt{C'}}{n}\sin n(x-l) \quad . (62).$$

and

$$-A \text{ tang } i-B'=\sqrt{C'}\cos n(x-l)$$

Equating A tang i for $x=a$, from 1st and 2d and $\sqrt{C'}$ is determined, its value being

$$\sqrt{C'}=\frac{B-B'-B\cos na+nF\sin na}{\cos n(a-l)}$$

But in the present case P is as yet an unknown quantity, from the fact that Q is partly supported by the stiffness of the beam at A, and partly at B. We may, however, eliminate P by aid of the condition that the ordinates y from 1st and 2d are equal at $x=a$.

Equating the ordinates y, we find, after much reduction,

$$\frac{P}{Q}=\frac{\sin n(l-a)+na\cos nl-\sin nl}{nl\cos nl-\sin nl}=q$$

$$r\ P=Qq. \qquad \ldots (63).$$

If $a = l$, $P = Q$; and if $a = o$, $P = o$; which are obviously correct.

The equation of the curve of the axis of the beam for AC, also the moment of flexure, will be

$$-Ty - Q\Big(q(l-x) - (a-x)\Big) =$$

$$-Q\Big((ql-a)\cos nx - \frac{q-1}{n}\sin nx\Big) \quad (64).$$

Similarly we have for BC

$$-'1\,y - Qq(l-x) =$$

$$-\frac{Q}{n}\Big(1-(1-q)\cos na + n(ql-a)\sin na\Big)$$

$$\frac{\sin n(l-x)}{\cos n(l-a)} \quad . \quad . \quad . \quad (65).$$

An inspection of the figure shows that there will always be a point of contra flexure in the beam, or where the moment of flexure will be zero.

Trying (64) for this point, placing the 2d number $= o$, we obtain

$$(1-q)\text{tang}\, nx = n(a-ql)$$

or

$$\text{tang}\, nx = na - \frac{\left(1-q\frac{l}{a}\right)}{1-q}$$

in which x locates the point of centro-flexure, and a the load. There is most doubt as to the relation of x and a when these quantities are small. When very small the tangent approximates the arc, when the fraction to na is less than 1, which makes x less than a.

Trying (65), the 2d member $=o$ when sin $n(l-x)=o$, or $n(l-x)=o$, π, 2π, &c. But $x=l$ when $n(l-x)=o$

$$x=l-\frac{\pi}{n} \text{ when } n(l-x)=\pi$$

But according to eq. (59) and Fig. 12, x in (65) has no value within the limits of the beam for a varying from o to l.

Hence we conclude that the point of contrary flexure is always between the load Q, and the fixed end of the beam.

The point of maximum moment is to be found where the 2d member of (65) is a maximum. This occurs where

$$\text{sin } n(l-x)=1 \text{ or } n(l-x)=\frac{\pi}{2}$$

$$\therefore \qquad x=l-\frac{\pi}{2n} \quad . \quad . \quad . \quad (60)$$

if we ignore a. But considering a, it appears that the max. moment of flexure is always at Q, except when a is $=$, or less than x in (60); conditions identical with case X, as regards the one point H, Fig. 12.

Either member of (64) makes the moment at A where x and $y=o$

$$=-Q(ql-a) \quad . \quad . \quad . \quad (66).$$

But where $x=a$, the other limit of x for this expression, the moment is

$$-Ty-Qq(l-a)=$$

$$-\frac{Q}{n}\Big(n(ql-a)\cos na-(q-1)\sin na \quad (67).$$

and it is difficult to tell which is the greater, except in particular cases.

An inspection of figure 13 shows that if the compression T can be great enough to hold the beam in a curve while Q is zero, the force P will be reversed. Now, if Q be given some real positive value increasing from zero up, it will eventually make P $=o$. But by (63), when P$=o$, $q=o$. Hence to find the relation between T and Q for this, make q$=$o in *(64) and (65.)*

XII. *Take the conditions as in* XI. *except change* T *to tension.*

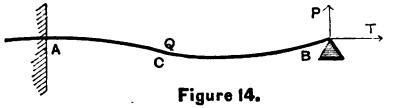


Figure 14.

1st. For the part AC, origin of co-ordinates at A, and y positive downward,

$$\varepsilon I\frac{d^2y}{dx^2}=+Ty+Q(a-x)-P(l-x)$$
$$=Py-(Q-P)x+(Qa-Pl)$$

Integrating by aid of (13) and determining the constants by the conditions $x=o$, for $y=o$, and $\frac{dy}{dx}=o$ for $x=o$ we get an equation of the curve of the axis of the beam AC, and also the moment of flexure

$$Ty-(Q-P)x-(Qa-Pl)=$$
$$\frac{Qa-Pl}{2}\left(e^{nx}+e^{-nx}\right)-$$
$$\frac{Q-P}{2n}\left(e^{nx}-e^{-nx}\right)=M \quad \ldots \quad (68).$$

But P is here an unknown quantity, and is to be found as it was in XI.

2d. For BC

$$\varepsilon 1\frac{d^2y}{dx^2}=\mathrm{T}y-\mathrm{P}(l-x)$$
$$=\mathrm{T}y+\mathrm{P}x-\mathrm{P}l$$

which may be integrated by (13), for $x=l$, $y=o$; giving value to one constant. Then place tang i, for the point C equal the same from (68), making known the second constant. Then equate the ordinates y for the point C, by which to determine P. The work is too tedious to detail here, but the result is

$$\frac{\mathrm{P}}{\mathrm{Q}}=\frac{e^{-nl}-e^{nl}-e^{-n(l-a)}+e^{n(l-a)}+na\left(e^{nl}+e^{-nl}\right)}{e^{-nl}-e^{nl}+nl\left(e^{nl}+e^{-nl}\right)} \quad . \quad . \quad . \quad (69).$$

When $a=l$ P=Q, and when $a=o$, P=0; which are correct for these points and verify the result.

The maximum moment of strain for the whole beam, and under all conditions is *probably at A*, as it is at this point for

the case that $T=o$. The probability seems to become a certainty from the fact that when T exists, its moment, which is counter to that of Q, is greatest where the deflection is greatest, or at intermediate points.

XIII. *Let the beam be fixed at both ends, be under tension* T *and have a load Q at the middle, and a uniformly distributed load.*

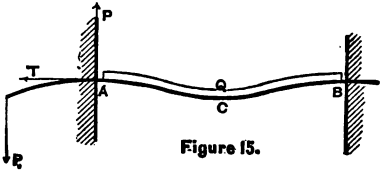


Figure 15.

$$\varepsilon I\frac{d^2y}{dx^2}=Ty+\frac{wx^2}{2}-Px+P_1(r+x)$$

$$=Ty+\frac{wx^2}{2}+(P_1-P)x+P_1r$$

where $P=P_1+\frac{wl}{2}+\frac{Q}{2}$

Integrating by aid of (13), and observ-

ing that for $x=o$, $y=o$, and that $\frac{dy}{dx}=o$, for $x-o$ and also for $x=\frac{l}{2}$; and we get, after reduction, the general equation of moments and of the axis of beam

$$Ty+\frac{wx^2}{2}-\frac{wl+Q}{2}x+P_1r=\frac{wl+Q}{4n}$$

$$\left(\begin{matrix} -nx & nx & 2-\frac{2wl}{wl+Q-e^{\frac{nl}{2}}-e^{-n\frac{l}{2}}} \\ e & -e & \end{matrix}\right)\left(\begin{matrix} nx & -nx \\ e & +e \end{matrix}\right)$$

$$-\frac{w}{n^2}=M \quad . \quad . \quad . \quad (70).$$

In case the load is only Q, make w $=o$, or if we have only the uniform load, make Q $=o$ and the resulting expressions will give the moment of flexure at any point in the length of the beam.

At the point A, the moment is a max. and found by making $x=o$, and $y=o$. Hence, as P_1r=moment at A, we have,

$$P_1 r = \frac{\frac{wl}{n}}{e^{\frac{l}{n\frac{1}{2}}} - e^{-n\frac{l}{2}}} + \frac{wl+Q}{2n} \cdot \frac{e^{\frac{l}{n\frac{1}{4}}} - e^{-n\frac{l}{4}}}{e^{\frac{l}{n\frac{1}{4}}} + e^{-n\frac{l}{4}}}$$

$$- \frac{w}{n^2} = M_{max} \quad . \quad . \quad (71).$$

When $w=o$, (71) becomes

$$P_1 r = \frac{Q}{2n} \cdot \frac{e^{\frac{l}{n\frac{1}{4}}} - e^{-n\frac{l}{4}}}{e^{\frac{l}{n\frac{1}{4}}} + e^{-n\frac{l}{4}}} = M_{max} \quad . \quad (72).$$

But when $Q=o$, (71) becomes

$$P_1 r = -\frac{wl}{2n} \cdot \frac{e^{n\frac{l}{2}} + e^{-n\frac{l}{2}}}{e^{n\frac{l}{2}} - e^{-n\frac{l}{2}}} - \frac{w}{n^2} = M_{max} \quad . \quad . \quad (73).$$

At the middle of the beam $x=\frac{l}{2}$ and the moment, for $w=o$, is the same as (72) except the sign is contrary.

But for $Q=o$; instead of (73) we have

$$\frac{\frac{wl}{n}}{e^{n\frac{l}{2}} - e^{-n\frac{l}{2}}} - \frac{w}{n^2} = M \quad . \quad . \quad . \quad (74).$$

The four last equations give the moments most likely to be needed, and the value of $P_1 r$ put in (70) gives us the equation of the curve of AQ and BQ is like it.

XIV. *Take the case as in* XIII., *except change* T *to compression.*

$$\varepsilon I\frac{d^2y}{dx^2}=-Ty-Px+\frac{wx^2}{2}+P_1(r+x)$$

$$=-Ty-(P-P_1)x+\frac{wx^2}{2}+P_1 r$$

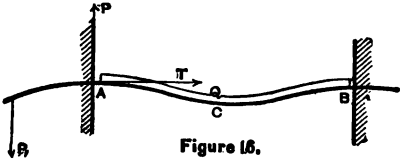


Figure 16.

Integrating by aid of (15), and observing that for $x=o$, $y=o$; and that $\frac{dy}{dx}=o$ for $x=o$ and for $x=\frac{l}{2}$; we obtain

$$-Ty-\frac{wl+Q}{2}x+\frac{w}{2}x^2+P_1 r=$$

$$= -\frac{wl+Q}{2n}\left(\sin nx + \cot n\frac{l}{2}\cos nx\right)$$

$$+\frac{Q}{2n}\cdot\frac{\cos nx}{\sin n\frac{l}{2}}+\frac{w}{n^2}=M \quad . \quad . \quad (75).$$

At A, x and $y=o$, and

$$P_1 r=-\frac{wl+Q}{2n}\cot n\frac{l}{2}+$$

$$\frac{Q}{2n}\cdot\frac{1}{\sin n\frac{l}{2}}+\frac{w}{n^2}=M_{max} \quad . \quad . \quad . \quad (76).$$

When $w=o$ (76), becomes

$$\frac{Q}{2n}\text{tang } n\frac{l}{4}=M_{max} \quad . \quad . \quad . \quad . \quad . \quad (77).$$

essentially the same as (58).

But when $Q=o$ in (76), we have

$$P_1 r=-\frac{wl}{2n}\cot n\frac{l}{2}+\frac{w}{n^2}=M_{max} \quad . \quad . \quad (78).$$

This is the same as at B, where $x=l$. Q still $=o$

At the middle of the beam $x=\frac{l}{2}$, and the moment for $w=o$, is the same as (77) except the sign is contrary. But for $Q=o$, instead of (78).

$$-\frac{wl}{2n \sin n\frac{l}{2}}+\frac{w}{n^2}=M \quad . \quad . \quad . \quad (79).$$

Thus the most needed moments are obtained, and the equation of the elastic axis of the beam is fully given, P_1r, in (75), being made known by (76).

Strength of Wrought-Iron Bridge Members.

II. — Practical Formulas for Beams, Struts, Columns and Semi-Columns.— Extended Comparison of Various Formulas with Experiment.

General Applicability.

All expressions for the moment M, from equation (17) to (79) inclusive are applicable in equations (6) and (8), the former giving the strain t_1, due to the bending moment alone, and the latter the total or max. strain t, in the section considered. Any section throughout the length of the beam may be examined for strain, but usually the section under maximum strain is the one to be attended to.

It is to be observed that the strain due to M, is not in any case of the existence of T to be taken as the only strain. That due to T *directly*, according to (7), is to

be included. For instance, points of contrary flexure are usually regarded as locating sections which are free from all strain except "transverse shear." But in all cases of points of contrary flexure mentioned above there will be not only transverse shear, but direct longitudinal tension or compression also. Hence the resultant stress will in these cases be diagonal in direction.

It is to be observed that T and t have been universally employed both for tension and compression. This was simply for convenience. In application, the proper distinction is to be made.

In cases where T is compressive in the above expressions for M, the beam may be regarded as acting partly like a beam and partly like a column, and we may vary the intensity of action from one almost purely of the first kind to one of the second kind.

ORDINARY BEAM FORMULAS

But at the limit $T=o$ some difficulty is experienced as, for instance (77) re-

duces to $\frac{0}{0}$ for T=o. But let the tangent be developed into a series, thus:

$$M_{max} = \frac{Q}{2n}\text{tang } n\frac{l}{4} =$$

$$\frac{Q}{2n}\left\{n\frac{l}{4}+\frac{n^3}{3}\left(\frac{l}{4}\right)^3+\frac{2n^5}{15}\left(\frac{l}{4}\right)^5+\&c\right\}$$

$$=\frac{Q}{2}\left\{\frac{l}{4}+\frac{n^2}{3}\left(\frac{l}{4}\right)^3+\&c\right\} \quad . \quad . \quad (80).$$

or

$$M_{max}=\frac{Ql}{8} \text{ for } n=o \quad . \quad . \quad . \quad (81).$$

This is the same expression for the maximum moment of flexure as we find given in works on the strength of beams for the case of a beam fixed at the ends and a load Q at the middle; conditions identical with (81) where T=o.

Similarly, (78), by expanding the $\cot n\frac{l}{2}$ gives

$$M_{max}=P_1r=-\frac{w}{n^2}+\frac{wl^2}{12}+\frac{wn^2l^4}{720}+\&c+\frac{w}{n^2}$$

$$=+\frac{wl^2}{12} \text{ for } n=o \quad . \quad . \quad . \quad (82).$$

which is also the same as given for this case.

Making use of the expanded series for e^x, expressions (72) and (73) will reduce also to (81) and (82). The other expressions of moment above may be likewise reduced for $T=o$, to the ordinary well-known formulas.

Thus it appears that the above equations and formulas are general, and the possibility of reducing the same to the ordinary expressions for simple cases corroborates all the circuitous mathematical work.

FORMULAS FOR COLUMNS, PILLARS, STRUTS, ETC.

In cases where the force T predominates, the piece acts mostly like a column. Passing to the limit by making the transverse forces zero, we have form (24):

$$\frac{-Ty_1 - Pl}{\text{tang } nl} = \frac{P}{n}$$

or

$$= \frac{Ty_1}{\text{tang } nl} = 0$$

if $P=o$. But as we propose that T shall be the predominating force applied to the piece, then $T y_1$ cannot be zero unless $y_1=o$, and the latter cannot be assured since T may produce considerable deflection to one side or the other. Theoretically there would be no deflection, but we know that practically there would be a very appreciable deflection for a piece of ten to twenty or more diameters in length before rupture; hence, we must treat the case as providing for a deflection, y_1.

Hence it appears that the denominator must be infinite, which requires that

$$nl=\frac{\pi}{2}$$

or, since $n^2=T\div\varepsilon I$

$$\frac{T}{\varepsilon I}=\frac{\pi^2}{4l^2} \quad . \quad . \quad . \quad (83).$$

for a column fixed at one end and free at the other, as in Fig. 17.

The same result is obtained from (31). This expression (83) is given in several works, and produced in different direct ways.

Equation (49), for $Q=o$, requires, for like reasons, that tang $n\frac{l}{2}=\infty$

or $$n\frac{l}{2}=\frac{\pi}{2}$$

$$\therefore \quad \frac{T}{\varepsilon I}=\frac{\pi^2}{l^2} \quad . \quad . \quad . \quad (84).$$

This expression is also obtained from (58), and is for columns having rounded

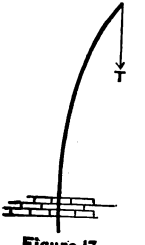


Figure 17.

ends; or, more specifically, pointed bearings. This expression is also given in several works on the strength of beams obtained usually by a direct process. By some the column is said to have rounded ends for this formula, while others do *not say* whether the ends should be

rounded, flat, or otherwise; that is, whether the end of the column is fixed or hinged, &c. The manner of holding the end, however, is not a matter of indifference, equation (49) showing that (84) is for rounded or hinged ends as shown in Fig. 18.

Figure 18.

Equation (77) or (78), for Q or $w=o$ and T not, requires that

$$\text{tang } n\frac{l}{4}=\infty \text{ or } n\frac{l}{4}=\frac{\pi}{2}$$

or that

$$\cot n\frac{l}{2}=\infty \text{ or } n\frac{l}{2}=\pi$$

either of which gives

$$\frac{T}{\varepsilon I}=\frac{4\pi^2}{l^2} \quad . \quad . \quad . \quad (85).$$

which is for columns fixed at both ends, or sometimes considered as having flat ends, as shown in Fig. 19.

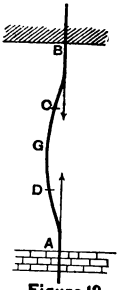


Figure 19.

This expression is not common in published works as applying to columns with fixed ends. An article of much merit which appeared in VAN NOSTRAND'S ENG. MAG., Vol. 17, p. 257, gives the above equation (84), but does not state whether the columns are for round or flat ends except as indicated by a figure like Fig. 18. But later articles in VAN NOSTRAND'S MAG. by J. D. Crehore and Prof. Baldwin give both (84) and (85,)

also (87) with statement of end conditions.

To obtain the expression for a column having one end fixed and the other end rounded, we naturally turn to case XI, above. But the deduction of a satisfactory expression from this is difficult. However, reasoning from analogy, compare Figs, 18 and 19 with their expressions. Thus,

$$\left.\begin{array}{l}\text{For Fig. 18, } nl = \pi \\ \text{For Fig. 19, } nl = 2\pi\end{array}\right\} \quad . \quad . \quad . \quad (86).$$

Observing that the curves are sinusoids--see eqs. (22) for $h=o$ and (47.)—we find AB, Fig. 18, like c D. Fig. 19, both being complete flattened sinusoids. But the middle of the beam at G, Fig. 19, is parallel to the same at A or B ; from which we conclude that

$$AD + BC + CD = 2 \text{ sinusoids.}$$

Hence the number of sinusoids in the curves Figs. 18 and 19 agree with the coefficients to π in the respective expressions (86).

Now for a column which has one end flat and the other rounded, the part CDA, Fig. 19, would represent it nearly, the chief point of difference being in the fact that C, Fig. 19, is not quite on the line of action AB. Assuming that the point D does not essentially change position as C is brought to the action line, then we can safely adopt $\frac{3}{2}$ as the coefficient to π for the expression corresponding to (86), because in Fig. 18 we have one sinusoid; in B, Fig. 19, two sinusoids, and in CDA $\frac{3}{2}$ sinusoids.

Now the point C, Fig. 19, is a point of contrary flexure, or a point where the bending moment is zero, and hence the condition of CDA would not be altered by cutting off the column at c and rounding it, thus giving, in CDA, a column with the desired conditions.

Hence the expression

$$nl=\frac{3}{2}\pi$$

or

$$\frac{T}{\varepsilon I}=\frac{9}{4}\frac{\pi^2}{l^2} \quad . \quad . \quad . \quad (87).$$

for the case of columns having one e
flat or fixed, and the other rounded
pointed.

The conditions are shown in Fig.

Figure 20.

where A is fixed, and B rounded
hinged.

These formulas, (83) and (84), h
been proposed for use by some write
without modification, though apparen
with much hesitation and doubt; oth
have declared them wholly unfit. It
pears that Hodgkinson compared (
in the form

$$T=\varepsilon\pi^2\frac{I}{l^2}=\frac{\varepsilon\pi^2}{64}\frac{d^4}{l^2}$$

where

$$I=\frac{\pi d^4}{64}$$

with his experimental results, but finding it discordant, only used it as suggestive of form for an expression for columns, thus:

$$W=C\frac{d^n}{l^m} \quad . \quad . \quad . \quad . \quad (88).$$

where W = wt. sustained, d = diameter, l length; and with C, m and n, constants to be determined by experiment. The differences made manifest by Hodgkinson's experiments between (84) and (88), as to the values of m and n, are such, except by further inquiry, as to destroy all confidence in (84) as a formula to use directly in calculating the strength of columns. The same is true of (85) and (87.) This is unfortunate as regards one important fact, viz: (84) contains the moment of inertia of the cross section instead of merely the diameter, so that *(84) may be* applied to any form of sec-

tion, such as round, square, oblong, I and H sections &c., while (88) must be prepared for each characteristic section. Also, (84) contains one important factor, ε, which qualifies that formula for at least this one property of the material going into the column. But evidently there is, at least, one other property which is equally important, viz: the crushing resistance; or, if preferred, the elastic limit, neither of which appears in (84). Hence, though (84) has some valuable features, it must be pronounced sadly deficient as a general formula, except possibly for very long columns, such as would fail by simply springing out, without breaking. But evidently, columns relatively short, though longer than "blocks," and such as would fail partly by crushing and partly by springing, should have a formula containing both the coefficient of elasticity ε, and some modulus of resistance, such as t, for compression or tension, usually the former. Again still shorter columns or "blocks" would only require the *modulus of crushing resistance*.

Let Fig. 21 represent a portion of a column under strain from the action of T, the strain being partly compressive and partly flexural.

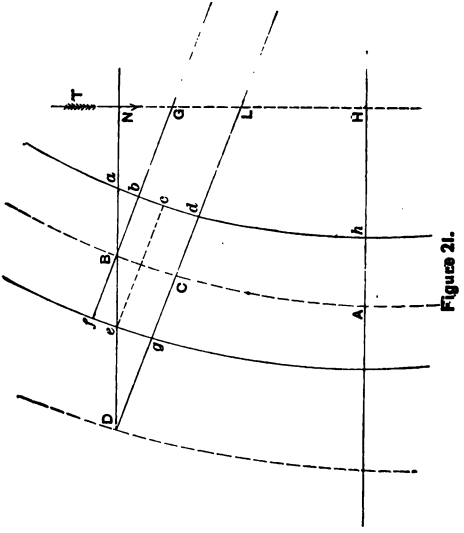


Figure 21.

To formulate the relations between these actions, suppose AB, Fig. 21, to be *a unit's* portion of a column near its

middle, which is sprung out of its original axial position NGH, to AC, the original axial fiber HN being now shortened and deflected to AC. The section *dg* was before flexure on the line *ae*, the intersection at D being a point on the real neutral axis of the column. The distance along DC to the intersection with AH, is the radius of curvature ρ. Draw BG parallel to DC, intersecting AH. The distance from this latter point to B is equal to ρ, because of the parallelism of BG and DC.

Then, according to (5), taking the line GH as the axial line of the column previous to flexure, and the line along which the compressive force T still acts, we will have the applied bending moment equal

$$\frac{\varepsilon I_1}{\rho} = T y_1 \quad . \quad . \quad . \quad (89).$$

where I is the principal moment of inertia, or moment of intertia, for an axis perpendicular to the plane of the curve ACB, at the center of gravity B of the section considered; y_1=BN, the moment

arm of T, and ε, the coefficient of elasticity.

This supposes that the ends of the column remain in the same position while the body deflects to one side or the other, as it will do in columns of considerable length before rupture—a fact known by experience. Hence the column is represented as deflected to one side in order that the conditions of the problem may coincide with experience. Equation (89) applies evidently to any unit length of the column if y_1 be supposed to be the ordinate to the axis for that point, and this without regard to conditions outside of that unit portion. It applies whether the column be fixed, rounded or otherwise, at the end, or even if the column be bellied. But the ordinate y_1 will be different for intermediate portions as depending on form of body of column as to taper, belly, &c., from which it appears that a bellied column will support differently from a prismatic one. Our analysis applies to the latter. The ordinate is marked y_1 instead of y, to dis-

tinguish it as the maximum ordinate to the axial curve of the column; that ordinate being chosen because the moment of T is here greatest, and hence may denote the ordinate at the rupturing point of the column.

In (89) the 2d member is the applied moment, and is that which simply bends the column. The 1st member is the moment of resistance to this bending, and represented in the figure by the triangle B*ab* of compression, and B*ef* of tension.

To obtain another equation take the origin of moments at D on the real neutral axis. Then the total moment of resistance is

$$\frac{\varepsilon I}{\rho} = T\,(y_1 + BD) \quad . \quad . \quad . \quad (90).$$

where I is now the moment of inertia of the section at B for an axis at D; $(y_1 + BD)$ the moment arm of T; and ρ the same as in (89), as already stated. By a well-known formula, we have

$$I = I_1 + K.\overline{BD}^2$$

B being the center of gravity of the section.

Introducing I, we have from (90),

$$\frac{\varepsilon I}{\rho}=\frac{\varepsilon I'}{\rho}+\frac{\varepsilon}{\rho}K\overline{BD}^2=Ty_1+T.BD$$

or, by (89)

$$\frac{\varepsilon}{\rho}K\overline{BD}^2=T.BD \quad . \quad . \quad . \quad (91).$$

But calling the shortening $BC=\lambda$, we have

$$\lambda : BD :: 1 : \rho$$

whence

$$\frac{BD}{\rho}=\lambda$$

and (91) becomes

$$\varepsilon K\lambda.BD=T.BD \quad . \quad . \quad . \quad (92).$$

But from the fundamental law of elasticity, see eq. ($4\frac{1}{2}$), observing that the shortening $BC=\lambda$ is for a unit length

$$T=\varepsilon K\lambda.$$

Combining this with (92), it becomes an identical equation which proves its correctness.

To determine the practical limitations

of the ordinate y_1 let T be applied at E, Fig. 22, to a prism of unit's length. First, let the distance, BE, = the *principal radius of gyration*, $= k$. The point B is the center of gravity of the section at ED,

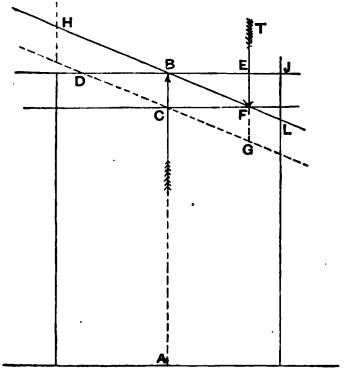


Figure 22.

or top of the prism. Now, as T is applied at E, let an equal and opposite force be applied at B, as shown. We thus have a couple acting to turn the top surface of *the prism* around B from the line EBD to

FBH. If now we remove the counteracting force at B, while T is kept constant, the point, B, will fall by the compressive action of T to some point C. Call BC$=\lambda$. This will be found $=$EF, when BE$=k$. For we have, by taking moments at B,

$$Tk=\frac{\varepsilon I_1}{\rho};\ \text{also } I_1=Kk^2, \text{ and } T=\varepsilon K\lambda$$

combining,

$$\frac{T}{\lambda}\frac{k^2}{\rho}=Tk \text{ or } k=\rho\lambda$$

whence

$$\rho : 1 :: k : \lambda, \text{ or : BC}$$

Hence $\lambda=$BC$=$EF as proposed.

Applying T at any other point, we have a different result, for suppose it at J. The compressive effect, BC, will be the same for the same value of T as long as it is the only compressive force. But the fall due to bending will be greater, even greater than JL, because the moment arm is greater.

Hence when a compressive force, T, is applied at the distance, k, from the center of the cross section of a prism, the

displacement, EF, due to bending is exactly equal to the displacement, FG, or BC, due to direct compression.

Next suppose the neutral axis to be taken at a distance qk from B while T is applied at E. Then eq. (90) gives

$$T(k+qk)=\frac{\varepsilon I}{\rho}=\frac{\varepsilon I_1}{\rho}+\frac{\varepsilon}{\rho}K(qk)^2$$

or since $Tk=\frac{\varepsilon I_1}{\rho}$ . . . (92a).

$$Tqk=\frac{\varepsilon K}{\rho}q^2k^2$$

$$Tk=\frac{\varepsilon K}{\rho}qk^2 <= \text{ or } > \frac{T}{\rho\lambda}k.qk$$

where $\rho\lambda=k$ when T is applied at E, and $T=\varepsilon K\lambda$

Hence

Tk=applied moment

<=or> moment of resistance.

To make the first and second members equal, we must put $q=1$; and hence we must also have the neutral axis at a distance, k, from the center, B, of the section, for the condition that the column is indifferent as to whether it deflects more

or less. If it deflects more, y_1 or BE, becomes $>k$, and BD $<k$, or $q<1$, and then the greater the displacement the greater is BE $=y_1$, and hence the greater the deflection of the column, Fig. 21, beyond $y_1=\text{BE}=k$, the greater is its tendency to deflect, because the applied moment becomes greater than the moment of resistance. But, on the other hand, when the ordinate, $y_1=$BE is less than k, BD is greater than k, $q>1$, and the column is stable. It appears, therefore, that Tk is the greatest admissible applied moment. We, therefore, follow the condition that in eqs. (90) to (92), BD $=y_1$.

Now in Fig. 21, we have

$$ad : (\text{BD}+d_1) :: 1 : \rho$$

or
$$ad=\frac{y_1+d_1}{\rho}$$

and
$$\varepsilon.ad=$$

where BD $=y_1$ $d_1=$Ba$=$distance from the center of gravity of the section to where the fiber ruptures by compression, and t the greatest admissible compressive force per square inch for a unit's length *of prism.*

$$\therefore \qquad ad=\frac{t}{\varepsilon}=\frac{y_1+d_1}{\rho} \quad . \quad . \quad . \quad (93).$$

But combining

$$\frac{\varepsilon \mathrm{I}_1}{\rho}=\mathrm{T}y_1 \quad . \quad . \quad . \quad (94).$$

with equations (84), (85), and (87), observing that I is the same as I_1 in (94), we get for columns

$$\left.\begin{array}{ll} \text{with round ends...} & \frac{1}{\rho}=y_1\frac{\pi}{l^2} \\ \text{with flat ends.....} & \frac{1}{\rho}=4y_1\frac{\pi^2}{l^2} \\ \text{with round and flat.} & \frac{1}{\rho}=\frac{9}{4}y_1\frac{\pi^2}{l^2} \\ \text{with fixed and free.} & \frac{1}{\rho}=\frac{1}{4}y_1\frac{\pi^2}{l^2} \end{array}\right\} \quad . \quad . \quad (95).$$

Combining these with (93), we obtain

$$\text{for round ends} \frac{tl^2}{\varepsilon\pi^2}=y_1^2+d_1y_1$$

or, solving for y_1

$$y_1=\frac{d_1}{2}\left(\sqrt{1+\frac{4tl^2}{\varepsilon d_1^2\pi^2}}-1\right) \quad . \quad (96).$$

But by eq. (8), the total compression on the critical fiber ah, Fig. 21, is

$$t=\mathrm{T}y_1\frac{d_1}{\mathrm{I}_1}+\frac{\mathrm{T}}{\mathrm{K}} \quad . \quad . \quad . \quad (96a).$$

or, introducing y_1 from (96)

$$t=\frac{T}{I_1}\frac{d_1^2}{2}\left(\sqrt{1+\frac{4tl^2}{\pi^2\varepsilon d_1^2}}-1\right)+\frac{T}{K}$$

and the remaining equations in (95) may be treated similarly.

Solving for the load, T, borne by the column, we obtain the following general formulæ, viz.: for columns with

Round ends

$$T=\frac{tK}{1+\frac{Kd_1^2}{2I_1}\left(\sqrt{1+\frac{4tl^2}{\pi^2\varepsilon d_1^2}}-1\right)} \quad . \quad (97).$$

Flat ends

$$T=\frac{tK}{1+\frac{Kd_1^2}{2I_1}\left(\sqrt{1+\frac{tl^2}{\pi^2\varepsilon d_1^2}}-1\right)} \quad . \quad (98).$$

Flat and round

$$T=\frac{tK}{1+\frac{Kd_1^2}{2I_1}\left(\sqrt{1+\frac{16}{9}\frac{tl^2}{\pi^2\varepsilon d_1^2}}-1\right)} \quad . \quad (99).$$

Fixed and free

$$T=\frac{tK}{1+\frac{Kd_1^2}{2I_1}\left(\sqrt{1+\frac{16tl^2}{\pi^2\varepsilon d_1^2}}-1\right)} \quad . \quad (100).$$

We observe that in using the equations (84), (85) and (87), our object is to obtain a relation between ρ and y_1, as given by the curves of sines of the elastic axes of the respective columns, and not to employ them as the foundations for the general formulas, because the latter object is found in (93), (94) and (96a), which give the limiting ordinate, y_1 the moment of flexure, and the limiting strain respectively.

In these formulas t is the greatest admissible compression per square inch, or the modulus of crushing, for breaking loads, K, the sectional area of the column at the middle, d_1 the distance from the center of gravity of the section to the fiber which ruptures first, I_1 the moment of inertia for an axis at the center of gravity of the section, and at right angles to the plane of the axial curve of the deflected column; l, the length of the column, and ε, the coefficient of elasticity of the material. Also we have

$$\frac{K}{I_1} = \frac{1}{k^2}$$

where k is the principal radius of gyration.

These formulas are readily adapted to special forms of section. For instance, for solid cylindrical columns with diameter$=2r$, (97) becomes, for rounded ends,

$$T=\frac{\pi r^2 t}{1+2\left(1+\frac{4tl^2}{\varepsilon r^2\pi^2}\right)^{\frac{1}{2}}-2}$$

But it is useless to make these adaptations since the present practice in designing bridge struts requires formulas which will be applicable to all conceivable forms of built struts or columns, such as combined I beams and channel bars, latticed channel bars, &c.

A Simplified or Approximate Expression.—In cases where the length of the column divided by the diameter is relatively small, the radical part of the denominator may be developed into a series, by aid of the binomial theorem, for a more convenient expression. In doing this, let us adopt one expression in common for (97), (98) and (99), by placing

the numerical coefficient under the radical equal to a, leaving the general expression for it to be determined later. Also, introducing the principal radius of gyration, k, we have

$$T=\frac{tK}{1+\frac{d_1^2}{2k^2}\left\{\left(1+\frac{atl^2}{\varepsilon d_1^3\pi^2}\right)^{\frac{1}{2}}-1\right\}} \quad . \quad (101).$$

Now by placing

$$v=\frac{atl^2}{4\pi^2\varepsilon k^2} \text{ and } z=\frac{k^2}{d_1^2} \quad . \quad . \quad . \quad (102).$$

we obtain, with the developed denominator,

$$T=\frac{tK}{1+v-v^2z+2v^3z^2-5v^4z^3+\&c} \quad . \quad (103).$$

Now, in examining this equation, we find that for very short columns, we only need to retain $1+v$, and the formula then appears independent of d_1, which indicates reasonably enough that a column failing almost entirely by crushing, might have d_1 reckoned in one direction about as well as in another, for the distance from the center of gravity of the failing cross section to the point of incipient

rupture; that is, for crushing on all sides the direction for d_1 is indeterminate, and d_1 as such, is not required.

For lengths somewhat greater, additional terms will be required for the denominator; but eventually, for considerably increased length, the denominator converges too slowly, requiring us to fall back upon the undeveloped denominator.

Special Simple Formula for Unusually Long Columns.—But at length we find a limit where the expression (101) will give the same value for the supporting power of the column as the corresponding simpler expression of (84), (85) or (87), which contain no factor for crushing resistance. Beyond this limit, the latter may be employed without hesitation, and failure will be expected to occur by springing instead of crushing.

To obtain an idea of the position of this limit, the ratio of length to diameter has been computed for solid cylindrical wrought iron columns and found to be for columns with

ends rounded ..... $\frac{l}{d}=36.2$

ends flat..... $\frac{l}{d}=72.4$

ends round and flat.... $\frac{l}{d}=54.6$

For square, hollow and open-built columns, the ratios are still greater.

Hence, these last named formulas, previously rejected, now return to us again for application to very long columns. For this purpose, fortunately, the objection to these formulas, previously stated, of the absence of the modulus of crushing, t, has no weight, because failure occurs by springing and not by crushing.

Criterion for Long Column Formula. —Evidently this limit is to be found by placing the values of the load T as given by the two formulas equal to each other, and then solving for the

$$\text{limiting ratio}\frac{l}{d}$$

For rounded ends we have therefore

$$T=\varepsilon I\frac{\pi^2}{l^2}=$$

$$\frac{tK}{1+\frac{Kd_1^2}{2I_1}\left(\sqrt{1+\frac{4tl^2}{\varepsilon\pi^2 d_1^2}}-1\right)} \quad . \quad (104).$$

To avoid multiplying the solutions, it is desirable to obtain one which includes the three conditions of ends rounded, ends flat, ends round and flat. This may be done by aid of a, which is $=4$ in (104). To adapt a coefficient for the first member we observe that $\frac{4}{a}$ is sufficient. To provide for the fact that d_1 and the radius of gyration k do not bear a constant ratio to each other, for all columns, assume

$$\frac{d_1}{k}=\beta$$

Then, observing that k^2 is equal, the moment of inertia divided by the section, we have by introducing a and β,

$$\frac{T}{K}=\frac{4\varepsilon\pi^2}{a\beta^2}\frac{d_1^2}{l^2}=$$

$$\frac{t}{1+\frac{\beta^2}{2}\sqrt{1+\frac{atl^2}{\varepsilon\pi^2 d_1^2}}-\frac{\beta^2}{2}} \quad . \quad (105).$$

an expression in which the first member, placed equal $\frac{T}{K}$, will give any one of the expressions (84), (85), or (87), by introducing proper values of α and β. Also the second member of which will give (97), (98) or (99), with like attention to α and β.

Clearing of fractions we obtain

$$\sqrt{1+\frac{atl^2}{\varepsilon\pi^2 d_1^2}}=\left(1-\frac{2}{\beta^2}\right)+\frac{atl^2}{2\varepsilon\pi^2 d_1^2}$$

Squaring, and solving for $\frac{l^2}{d_1^2}$, we obtain

$$\frac{l^2}{d_1^2}=\frac{4}{a\beta}\left(1+\frac{1}{\beta}\right)\frac{\varepsilon\pi^2}{t} \quad . \quad . \quad . \quad (106).$$

a formula which serves as the criterion of applicability for $\frac{T}{K}=$ first part; or= second part of (105), as a formula for extremely great or for ordinary lengths, respectively.

When the axis for the moment of inertia is an axis of symmetry for the cross section, we have $2d_1=d=$ whole diame-

ter of column. This is the thickness in this case in the direction of the plane of the curve of flexure of the column, and is the extreme thickness, whether the column be solid, hollow or open-built.

When the cross section is thus symmetrical, we have

$$\frac{l}{2d_1}=\frac{l}{d}=\pi\left\{\frac{1}{\alpha\beta}\left(1+\frac{1}{\beta}\right)\frac{\varepsilon}{t}\right\}^{\frac{1}{2}} \quad . \quad . \quad (107).$$

As an example, in cylindrical columns, d_1 is the radius of the column, and k the radius of gyration.

$$\frac{d}{k}=2=\beta$$

and for flat ends,

$$\alpha=1$$

Taking $\varepsilon=28{,}000{,}000$ and $t=40{,}000$, we obtain from (106) or (107) the value 72.4, the same as given above.

FORMULA FOR END CONDITIONS.

Let us now determine a general expression for the qualifying factor α, an expression which shall duly qualify the general column formulas (105) for the

case of fixed ends, flat ends, or of rounded ends (pointed); or for pin ends with large pins under friction, or for flat ends which are narrower than the middle section, or for one of these forms at one end, and another at the other, &c.

From (97), (98), and (99), we already have

$$\left.\begin{array}{l}\text{For pointed ends}\ldots.a=\ 4, m=o\\ \text{For fixed ends}\ldots\ldots a=\ 1, m=2\\ \text{For pointed and fixed.} a=\frac{4}{9}, m=1\end{array}\right\}\ (108)$$

where m stands for the moment effect of the end conditions preventing lateral displacement of column at the middle. In Fig. 23, the pointed ends a and b offer no hindrance to the side deflection of the middle, m, of the column, hence in (108) m is zero. But at c and d, there is the stiffeners of the fixed ends, to prevent the middle from deflecting; and hence for this case in (108), $m=2$, similarly for the third part of Fig. 23, and of (108) $m=1$.

Fixed Ends.—Now if we consider the stiffness at a fixed end as equivalent to a

moment $p\frac{l}{2}$ preventing lateral displacement of the middle, we may combine it

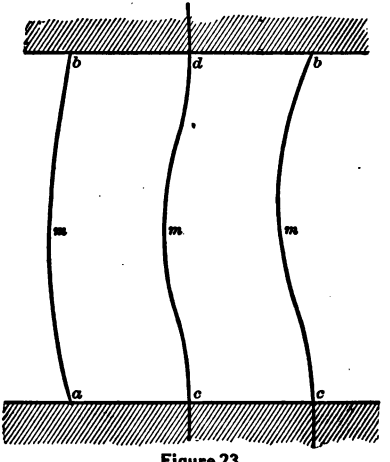


Figure 23.

with the end and middle moments due to T as follows:

The expression Tk, see (92a), appears to be the max. admissible moment of *flexure at the* middle. Using a similar *expression for* the end, we have

$$p\frac{l}{2}=\mathrm{T}k'$$

where k' is different from k when the fixed end of column differs in size and form from the middle section. Designating the latter, or radius of gyration of the middle section by k_m, and of the end surfaces or sections by k_c and k_d, we will have the total counteracting moment

$$\left(\frac{p\frac{l}{2}}{\mathrm{T}k_m}\right)=\frac{\mathrm{T}k_c+\mathrm{T}k_d}{\mathrm{T}k_m}=\frac{k_c+k_d}{k_m} \quad . \quad (109).$$

an expression which admits of any size of end sections less than the middle section. Even the ends c and d may differ. Now we find by trial that a may be obtained from the expression

$$a=\frac{4}{\left(1+\frac{k_c+k_d}{2k_m}\right)^2} \quad . \quad . \quad . \quad (110).$$

for ends actually fixed.

To illustrate, if k_c and k_d are zero, $a=4$ as for pointed ends, thus producing (97) from (105) as regards qualifying with respect to a. Again put $k_c=k_d=$

$= k_m$ and $a=1$, as for (98). Finally for $k_c = k_m$ and $k_d = o$, we have $a = \frac{1}{9}$ as required in (99).

But suppose the ends are only half as large as the middle and similar. Then $k_c = k_d = \frac{1}{2}k_m$ and a becomes $\frac{1}{9}$, the same as for one end pointed and the other as large as the middle, and fixed.

To distinguish between actual fixed ends, and flat ends, it appears that in the former the column cannot fail at the end without a similar failure at the middle. According to the conclusion BD$=y_1$ in Fig. 22, there must be some tension in the section both at the middle and end of column. That is; the neutral axis is at D when rupture takes place, and outside of D there will be tension. Hence when the end sections are cut as in flat ends; that tension cannot act, and the column is weaker than for actual fixed ends. It appears therefore that the neutral axis should not approach the axis of the flat ends nearer than the outermost surface of the column. As the neutral axis is at D, where BD=the radius of gyration, k;

we may obtain in flat ends, the equivalent to fixed ends, by making the radius of gyration of the flat end surface of the column equal to d_1. This necessitates enlarged, or disc-like ends.

Equation (110) is supposed to meet the real case of fixed ends. To adapt it to the case of flat ends, it is only necessary to replace k_m by d_1 : because the same value of a would then be given by (110) thus modified for k_c and $k_d = d_1$, as would now be given by (110), as above, for k_c and $k_d = k_m$. That would make the radius of gyration of the flat end bearing $= d_1$ and the neutral axis could not approach the axis at the end of the column nearer than d_1. Then there would be no tendency to tension in the flat ends, as well as no power to resist it.

Again in built columns it is often the case that thickening pieces are laid on at the ends, thus making the area of the flat ends greater than the sectional area at the middle. From analogy with the usual relation of k and K in sections

of beams, it appears that the radii of gyration k_c k_d and d_1, should each be multiplied by the square root of the proper section.

Hence, to include the two modifications of (110) now considered, it appears that the equation (110) should be written.

$$a=\frac{4}{\left(1+\frac{k_c\sqrt{K_c}+k_d\sqrt{K_d}}{2d_1\sqrt{K_m}}\right)^2} \qquad (110a).$$

The meaning of the subscripts is indicated in Fig. 23.

When the end surfaces or sections in every way equal the middle section of the column, the capitals, K, divide out and disappear, also when the ends are actually fixed d_1 should change to k_m and thence (110a) returns to (110).

When the flat ends are unenlarged, either in diameter or section, the capitals disappear; but d_1 remains and the denominator of (110a) is smaller than of (110), which makes the strength of columns with unenlarged flat ends less

than that for fixed ends, which is evidently as it should be.

In no case should (110a) ever be used when the denominator gives a value greater than the denominator of (110).

Pin Bearings.—The pin bearing for the end mounting of a column is usually considered as acting in effect like a knife-edge terminal to the column. But it is evident at once that the friction of the pin must cause the column to act as though the end had some breadth which is appreciable, and hence not to be ignored. To provide for this, let r be the radius of the pin, and f the coefficient of friction.

Then, as the column bows to one side or the other, the point of bearing of the hole upon the pin will be thrown to one side a distance ox Fig. 24, which distance is greater or less according to the value of f. Now as is well known, the value of ox is

$$ox = fr.$$

By Fig. 23, this may be designated as

fr_a or fr_b, as to whether it is expressed for one pin or the other.

In moment effect this is to be treated as an arm for T, so that the Tfr here, corresponds with Tk in respect to the fixed end. Hence, according to (109), we may write

$$\frac{\left(p_1\frac{l}{2}\right)}{\mathrm{T}k_m}=\frac{fr_a+fr_b}{k_m}$$

a quantity which is of the same kind as (109) and may be incorporated in (110a) thus

$$a=\frac{4}{\left(1+\frac{k_c\sqrt{\mathrm{K_c}}+k_d\sqrt{\mathrm{K_d}}}{2d_1\sqrt{\mathrm{K}_m}}+f\frac{r_a+r_b}{2k_m}\right)^2} \quad \ldots \quad (111).$$

In bridges, the pin is of considerable size, and the coefficient of friction quite high; probably between $f=.2$, to .3 for new bridges, and as high as .5 for older ones.

To illustrate, suppose the pin to be 4 inches in diameter at a and b. Then $fr_a = fr_b = .25\times 2''=.5''$, or$=\frac{1}{2}$ inch.

If the column be a solid cylinder, 8 inches in diameter at the middle $k_m = 2''$; and observing that k_c and k_d are both zero here, we have

$$a = 3.16$$

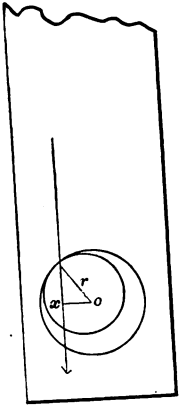


Figure 24.

As ordinarily treated, this would be taken as 4 instead of 3.16.

Relative Positions of Pin Bearings.— Now with respect to knife-edged vs. point-

ed ends, it is evident that when the knife edges are in one plane, and so oriented as to admit of deflection in the same direction as when the ends are pointed, the supporting power of the column is the same for either form of end. But when the knife edges are in a plane at right angles to the plane of deflection for points, the strength of the column is greater. This is with the understanding that the moment of inertia of the middle section is greater for an axis in one direction than in another. With a given cross-section of column the greatest strength with pin bearings is obtained by placing the knife edges parallel to the axis for that max. moment of inertia of that cross section. For deflections constrained by guides to be in the line of the knife edges, the strength is as for a column with flat ends. When a knife edge is placed at each end of a column, the strength is increased by placing the knife edges at right angles to each other. But for this hypothesis it is difficult to *calculate the* real strength of the col-

umn, or to determine the direction of the deflection.

To investigate this let us first take the first part of eq. (105). Replacing β by its value we have for the supporting power per square inch

$$\frac{T}{K}=\frac{4\varepsilon\pi^2}{a}\cdot\frac{k^2}{l^2}=t$$

Let l be constant for a particular case. Then the strength is constant if $\frac{k^2}{a}$ is constant. If this ratio is variable the strength of the column is variable, for deflection in different directions as controlled by guides near the middle. For each position of the guides there may be noted a deflection, a radius of gyration, and a knife edge component.

According to considerations previously noted, the column is counteracted in its tendency to deflect by a moment Tk at the middle, and Tx, say, at the end; where x is the arm of the moment. Now for the middle cross section of the column; let the radius of gyration be

found for a number of different directions within the circumference. Lay these off as in Fig. 25 where OE OG OH OK

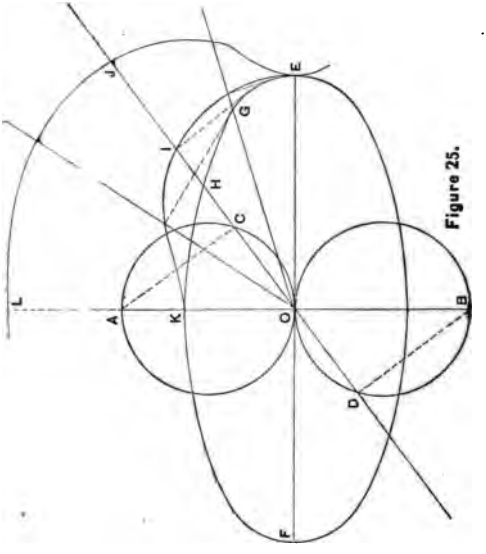


Figure 25.

are 4 values for the 4 respective directions. Then trace the curve EG HKE. *Any radius* vector to this curve will be

the value of k lying in the direction of that radius vector.

The point O represents the axis of the column. Let AB represent the knife edge bearing at one end of the column, its effective length being laid off to the same scale as k. The effective length should be less than the actual length, for the same reason that k is less than d_1; that is, the pin bearing will crush for some distance, corresponding to $d_1 - k$.

Now for a constrained deflection toward OA, lay off $x = OA$ as the moment arm. This added to k gives $OK + OA = OL$. Again for a constrained deflection toward OJ the knife-edge component of OA will be $OC = OA$ projected upon OJ. C is a point in a circle OCA giving all components. The curve EGHK should be treated in a similar way for finding the components. For instance project upon OJ by the tangent from the curve EGH giving I one point in the curve EIK. Any radius vector to this curve gives the k component for that direction. Now add OC and OI giving OJ.

J is another point in the curve EJL, a radius rector to which represents the resultant moment arm for the moment counteracting deflection in the direction of radius vector considered.

We should yet add the counteracting effect due to the opposite end of the column. Another curve outside of EJL including the three; the middle and the two ends. Then the minimum radius vector to that curve will be in the direction in which the column will deflect when unconstrained.

In practice it will be found that for almost every possible case, the minimum radius vector will lie in the direction of OA or OE so that the column will but rarely deflect diagonally. The minimum load value of the column will be found by computing for the direction of deflection above indicated. If the deflection takes in the direction of the pins, a is to be found from (111) by ignoring r and taking k_c and k_d as equal the respective values of OA.

If the knife edges be at right angles,

the curve, Fig. 25, obtained will be found to have minimum ordinates in the directions of the pins or knife edges.

Fig. 25 applies especially for the long column formula. For shorter columns proper to the formula (101) we will probably find less decided differences between max. and min. radii vectors to Fig. 25 until for short blocks the differences are either indeterminate or indistinguishable.

Open Column Stays.—Open built columns are occasionally deficient in trussing or latticing. For instance, two 8″ channel bars may be joined by $\frac{3}{8}$″ plates 10″ square riveted on in such a way that the plane sides of the bars are back to back with an intervening space of 6 or 7 inches, the flanges being turned outward. Then if the plates are placed 3′ 10″ apart along the bars, the column thus constructed will probably be deficient in cross stays. In such case the column will fail to carry the load determined for it by the usual formula and

actual moment of inertia of the cross section. To illustrate, the failure may be as shown in Fig. 26 as indeed experiment

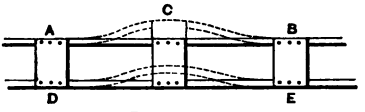


Figure 26.

has given, the parts ABC acting like little columns with fixed ends. It is proposed to find the length of a column made of the same plates fully stayed, which will be equivalent in strength to the actual part AB with its deficient stays. Let l = AB less the width of the stay C; l' = length of equivalent full-stayed column. Then if the parts AB and DE are regarded as small columns with fixed ends, we have by applying the simple long-column formula, for the purpose of securing a simple correction,

$$T = 4\varepsilon I' \frac{\pi^2}{l'^2} = 4\,\varepsilon I_1 \frac{\pi^2}{l'^2}$$

where I' is the sum obtained by finding

the least moment of inertia of each of the parts AB DE &c. taken separately and added; and I_1 the least moment of inertia of the entire section of the column obtained in the usual way as such. In this section, the staying is not counted. There the square of the ratio obtained by dividing the equivalent full-stayed column length by the actual column length supposed insufficiently stayed is obtained by solving the above for l''^2 and dividing by l^2, giving

$$\left(\frac{l''}{l}\right)^2 = \frac{I_1}{I'}\left(\frac{l'}{l}\right) \quad . \quad . \quad . \quad (112).$$

This forms a co-efficient for l^2 in any of the column formulas. For instance, if $\frac{I_1}{I'}\left(\frac{l'}{l}\right)^2 = 3$, then in formulas (97), (98), or (99), &c., l^2 therein, should be multiplied by 3, when (112) is less than 1 it is not to be applied. This fact indicates that the column is sufficiently stayed.

Stay Spacing.—In the last statement we have the suggestion of a formula for *determining* the spacing in built columns.

That is to say, placing the 2d member of (112)=1, and solving for l', we have twice the greatest allowable space between stay points in open-built columns, or,

Greatest admissible space between stay points

$$\frac{l'}{2}=\frac{l}{2}\sqrt{\frac{I'}{I_1}} \quad . \quad . \quad . \quad . \quad . \quad (113).$$

Thus, if a column be built of 4 angle bars L $\frac{1}{2}''\times 3''$, situated as forming the corners of a 12″ square in a column 20′ long, the greatest inter-stay space will be about 16 inches. As an example from actual experiment, column No. 33 of G. Bouscaren's report of experiments to the A. S. C. E. gives the spacing of the open column at 18 inches, and it failed by buckling between the lattices. The spacing, according to the formula (113), should not have exceeded about 14 inches, and hence it should have failed, as observed.

As to the space in open columns of *different* construction, it is evident that

$AB=l'$ is twice the space in Fig. 26, and in the first part of Fig. 27. But in the second part of the latter, $AB=l'$ is to be counted as the space where the cross and diagonal staying together serves to fix the ends of the parts, AB.

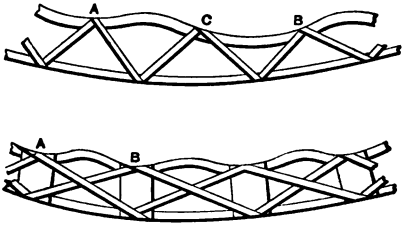


Figure 27.

PRACTICAL FORMULAS FOR COLUMNS.

Collecting the formulas which are useful in practice, we have:

For ordinary columns

$$\frac{T}{K}=\frac{t}{1+\frac{d_1^2}{2k^2}\left(\sqrt{1+\frac{atl^2}{\varepsilon\pi^2 d_1^2}}-1\right)} \quad . \quad . \quad (114).$$

For very long columns,

$$\frac{T}{K}=\frac{4}{a}\varepsilon\pi^2\frac{k^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad (115).$$

Criterion for applying 114 or 115

$$\frac{l}{2d_1}=\left\{\frac{\pi^2}{a\beta}\left(1+\frac{1}{\beta}\right)\frac{\varepsilon}{t}\right\}^{\frac{1}{2}} \quad . \quad . \quad (116).$$

$$\beta=\frac{d_1}{k}$$

For adapting to actual end conditions, see Fig. 23.

For flat ends,

$$a=\frac{4}{\left(1+\frac{k_c\sqrt{K_c}+k^d\sqrt{K_d}}{2d_1\sqrt{K_m}}+f\frac{r_a+r_b}{2k_m}\right)^2} \quad . \quad . \quad . \quad (117).$$

For *fixed* ends change d_1 to k_m

Co-efficient for l^2, in (114) and (115) for columns deficient in staying,

$$\left(\frac{l''}{l}\right)^2=\frac{I_1}{I'}\left(\frac{l'}{l}\right)^2 \quad . \quad . \quad . \quad (118).$$

Inter-stay spaces in open columns

$$\frac{l'}{2}=\frac{l}{2}\sqrt{\frac{I'}{I_1}} \quad . \quad . \quad . \quad (119).$$

where $l'=AB$ in Figs. 26 and 27.

In these formulas,

T = load borne by the column.

t = compressive resistance per square inch corresponding to T.

K = cross section of column, K_m at middle; K_c and K_d at ends, either when flat or fixed, and when flat to be taken as the flat-end surface. Only for suporting members.

d_1 = distance from the center of gravity of the section K_m to where the strain is t, and measured on the plane of the curve of flexure.

$d = 2d_1$ in symmetrical cross sections, and equal the diameter of the column reckoned parallel to plane of flexure.

k = radius of gyration of the cross section, k_m, least radius at the middle of column; k_c and k_d radii for the end sections for fixed ends; or of the end surface for flat ends, and to include supporting members only.

l = length of the column.

ε = coefficient of elasticity of the material composing the column.

f = coefficient of friction of a column on its pin bearing, = about .25 to .5.

r_a = radius of pin at one end of column.

r_b = radius of pin at the other end of column.

I_1 = least moment of inertia of the entire cross section of supporting members taken in their actual relative position, and exclusive of staying.

I' = aggregate moment of inertia obtained by finding the least moment of inertia of supporting members taken separately, and adding

$\frac{l}{2d_1}$ > than in (116), apply (115).

$\frac{l}{2d_1}$ < than in (116), apply (114).

$a = 4$ for columns terminated in points or knife edges.

$a = 1$ for columns terminated in fixed ends no smaller than the middle of column.

$a = \frac{16}{9}$ for columns having one end pointed and one end fixed and large as middle.

a = (117) in other cases.

OTHER FORMULAS.

The well-known formulas of Hodgkinson appear to be the first of substantial value, their constants being wholly empirical. They were applicable to only a few forms of cross section, and entirely inadequate to the present engineering practice.

Lewis Gordon determined the constants for Tredgold's formula* from the

* The formula of Tredgold is given in Tredgold on the *Strength of Cast Iron and Other Metals*, 2d ed., page 183, formula XIV.; and is

$$w = \frac{15300\, bd^3}{d^2 + 18\, l^2},$$

which is readily written in the form

$$w = \frac{15300\, bd}{1 + 18\frac{l^2}{d^2}},$$

where bd is the section of the column, =S, or K; and l the length.

Tredgold gives credit for mathematical work to Dr. Young.

This is precisely the formula usually called Gordon's, but it appears to be due to other authors, even to the determination of the empirical constants for at least one case, probably cast iron, as appears from the formula as just quoted. But there can be no doubt of Gordon having determined the constants for this formula for a variety of materials, though there may be reasonable doubts of his being justly entitled to the credit usually accorded to him for the formula as a whole, and much more so for Rankine's modification of it where the radius of gyration is introduced.

experiments of Hodgkinson, which, on account of its partly theoretical basis, was found to have a much wider range of applicability than Hodgkinson's formula. It came into quite general use, and was known as Gordon's formula. Rankine has displaced the least diameter, h, in this formula by the least radius of gyration, thus producing the most acceptable formula in use to-day.

The form of this expression is:

$$w = \frac{f\mathrm{K}}{1 + a\frac{l^2}{k^2}} \quad . \quad . \quad . \quad . \quad (120).$$

where f and a are empirical constants, the former being nearly the ultimate resistance to crushing; K and k, the cross section and its least radius of gyration respectively, at the middle of the column length, l. The values adopted by different engineers for f and of a differ greatly, even for a given material, so that in bridge specifications it is common to see the chosen values of f and a given. Besides this, it is impossible to determine values

for f and a for any particular material, except by actual column experiments—a fact which practically limits the formula to a few materials, and even then to a comparatively few experiments. Hence, a general formula whose constants consist of such factors as the coefficient of elasticity, modulus of crushing, &c., seems, by reason of the plentitude of these factors, very desirable.

These facts have caused search for such theoretical formula. One of the first formulas of this kind appears to have been proposed by Reuleaux, and published in *Der Constructeur*, the same being quoted by Ritter in his translation by Sankey of *Bridges and Roofs*, in 1879. See *Sankey's Ritter*, p. 345. The first edition was published in 1862.

This expression for rounded ends is of the form

$$w = \frac{c\mathrm{K}}{1+\frac{c\mathrm{K}l^2}{8\varepsilon \mathrm{I}}} = \frac{c\mathrm{K}}{1+\frac{c}{8\varepsilon}\frac{l^2}{k^2}} \quad . \quad . \quad . \quad (121).$$

where c = resistance to compression, proposed to be taken within the elastic limit for a safe load, ε the coefficient of elasticity. K l and k as before. The form of (121) is seen to be the same as (120). That is to say, the load for a given material is equal to a constant times the section; divided by 1 + a constant times the square of the ratio of the length to the radius of gyration of cross section

Notwithstanding the agreement in form, (121) has the advantage of applicability to all materials for which C and ε are known, for many of which (120) might not be known. Even if (121) should give discrepancies as it stands, an empirical qualifying coefficient could be found which would probably be nearly the same for one material as another.

An expression of the same form as (121) for round ends was proposed by Mr. J. D. Crehore, and published in VAN NOSTRANDS MAG. for Dec., 1879, under the title, "A New Rational Formula for Pillars." Mr. Crehore's formula however differed

in displacing 8 by π^2 in the denominator of (121). To account for this, Reuleaux treated the axis of the deflected column as a circle arc, while Crehore showed it to be a sinusoid. Hence Crehore's formula for rounded ends was

$$w = \frac{CK}{1 + \frac{Cl^2}{\pi^2 \varepsilon k^2}} \quad . \quad . \quad . \quad (122).$$

An expression identical with this for columns with round ends was given in an article by E. Hatzel, translated for VAN NOSTRAND'S ENG. MAG. for Sept., 1877; except in Hatzel's formula the moment of inertia was taken for an axis at other points in the section than the center of gravity; whence the radius of gyration k would be other than the principal radius.

Also Crehore showed that for flat ends the 2d term of the denominator of (122) should be divided by 4, and for one flat and one rounded end it should be divided by $\frac{1}{2}\frac{6}{9}$.

Crehore's formulas were reproduced

in an elaborate article by Professor W. Baldwin, and published in VAN NOSTRAND'S MAG. for May, 1880, with due credit to Mr. Crehore. But in the same article Professor Baldwin utterly abandons these formulas, and proceeds to the discussion of the Gordon formula containing the least thickness h, and of the Rankine formula containing the least radius of gyration k. Adopting the latter, its constants are determined from a great number of experiments cited, and for a variety of materials. But no attempt was made to generalize by retaining C and ε. The reason given is that ε is not constant in failing material, and at the point of rupture is unknown. This is to be admitted, especially in tension; but in compression the value remains very nearly constant for both cast and wrought iron. This is fortunate, because the chief factor concerned in the strength of columns is the compressive resistance. As the column fails by deflecting to one side, the material is most severely compressed at the concave side, and only a

small portion of the cross section will be involved in incipient fracture. Existing results of experiment show that ε is practicably constant for compression of cast and wrought iron to near the failing point, where it drops suddenly from 10 to 20 per cent. Hence if a tenth of the section is effected by 15 per cent. reduction in ε, the resulting effect upon the whole section would be one and one-half per cent., a quantity which may be ignored in calculating the strength of columns. Hence we propose to compare the rational formulas with experiment before giving them up.

But before taking up numerical facts we observe that the Reuleaux, Hatzel, or Crehore formula differ from the rational formula (97) for rounded ends, believed to be given for the first time in this article, in the form of the denominator. To show that this difference is an important one, we have from Fig. 21

$$ab : bB :: BA : \rho$$

or

$$ab : d_1 :: 1 : \rho$$

Also ($41\frac{1}{2}$) gives $t_1 = \varepsilon . ab$

$$\therefore \qquad \frac{1}{\rho} = \frac{ab}{d_1} = \frac{t_1}{\varepsilon d_1} = y_1 \frac{\pi^2}{l^2}$$

the last expression being found in the first of (95).
whence

$$y_1 = \frac{t_1 l^2}{\varepsilon d_1 \pi^2}$$

in which t_1 is the compressive force which is sufficient to cause the shortening ab and a force which is due to bending only. Combining this with (8) or (96 a) we have

$$t = \frac{\mathrm{T} t_1 l^2}{\varepsilon \mathrm{I}_1 \pi^2} + \frac{\mathrm{T}}{\mathrm{K}}$$

which, solved for T, gives for columns with rounded ends,

$$\mathrm{T} = \frac{t\mathrm{K}}{1 + \frac{t_1 \mathrm{K} l^2}{\pi^2 \varepsilon \mathrm{I}_1}}$$

$$= \frac{t\mathrm{K}}{1 + \frac{t_1 l_2}{\pi^2 \varepsilon k^2}} \qquad . \quad . \quad . \quad (123).$$

an expression identical with Crehore's, ex-

cept that t_1 is not the compressive resistance; Fig. 21 showing that it is only a part of it. In this expression t is the total compression represented by ad, and hence for rupture the total compression t=C of Crehore's formula. But t_1, represented by ab, cannot be C, and hence some oversight appears to have been committed in making it C in formula (122). This explains the difference in form between (97) and (122); and gives sufficient cause for a disagreement of one or the other with experimental results.

In the developed denominator of (97) see (102) and (103), we have in the terms, $1+v$, (observing that a=4 for rounded ends) the identical denominator of (122). Hence to the remaining terms of the denominator of (103) we may look for the compensating quantity for the error C=t_1. The same fault is common to the Reuleaux, and to the Hatzel formulas though in the former a compensating effect appears to be provided in the values compounded of C and ε for use; and in

the latter in assuming the moment of inertia axis at or near the convex side of column.

Direct Comparison.—But the column formula problem is one which cannot be satisfactorily settled for practical men from a purely theoretical standpoint. I therefore offer the following comparison of column formulas. As the numerator may be considered as practically the same in all, the denominators only are compared.

Hence for the denominator we have for the

Gordon formula, $1+a\frac{l^2}{h^2}$

For round ends.	Round and fixed.	Fixed.
$a=\frac{4}{3000}$	$\frac{2}{3000}$	$\frac{1}{3000}$

Rankine formula, $1+b\frac{l^2}{k^2}$

For round ends.	Round and fixed.	Fixed.
$b=\frac{4}{36000}$	$\frac{16}{9}\ \frac{1}{36000}$	$\frac{1}{36000}$

By G. Bouscaren, C.E., and adopted by Keystone Bridge Company.

For round ends.	Round and fixed.	Fixed.
$b=\frac{2}{36000}$	$\frac{3}{2}\ \frac{1}{36000}$	$\frac{1}{36000}$

Reuleaux formula, $1+\frac{Cl^2}{8\varepsilon k^2}$ round ends

Crehore formula, $1+q\frac{Cl^2}{\pi^2\varepsilon k^2}$

For round ends.	Round and fixed.	Fixed.
$q=\ 1$	$\frac{1}{2.28}$	$\frac{1}{4}$

New formula, $1+\frac{d_1^2}{2k^2}\left(\sqrt{1+\frac{atl^2}{\pi^2\varepsilon d_1^3}}-1\right)$

For round ends.	Round and fixed.	Fixed.
$a=\ 4$	$\frac{16}{9}$	1

From these expressions for the denominators, I have computed the results *found in* the following

TABLE OF VALUES OF DENOMINATORS FOR WROUGHT IRON.

Data.	End.	Gordon.	Rankine.	Bouscaren-Keystone.	Crehore.	New.
Solid cylinder.....	Rounded.....	5.8	7.4	4.2	9.23	5.20
l=10 ft.....	Round and flat.	3.4	3.84	3.4	4.66	3.74
Diam.=2 in.....	Flat.....	2.2	2.6	2.6	3.06	2.50
Bar 2×12 in.....	Rounded.....	5.8	5.8	3 4	7.19	4.06
	Round and flat.	3.4	3.13	2.8	3.75	3.06
l=10 ft.....	Flat.....	2.2	2.2	2.2	2.54	2.17
Hollow cylinder...	Rounded.....	4.00	1.26	1.13	1.33	1.30
Diam. 8 in. and 6 in.	Round and flat.	2.50	1.14	1.10	1.15	1.17
l=10 ft..... ...	Flat.....	1.75	1.06	1.06	1.08	1.08
Hollow Rectangle..	Rounded	3.70	2.49	1.74	2 92	2.16
8×15 in. ×6×13 in,	Round and flat.	2.35	1.65	1.56	1.85	1.74
l=30 ft.....	Flat.....	1.67	1.37	1.37	1.48	1.39
Means.....		3.231	2.828	2.213	3.353	2.464

For the last two columns $t=C=40000$, $\varepsilon=23370000$, and $\frac{t}{\varepsilon\pi^2}=\frac{1}{7000}$.

The first three formulas named in the table have been in quite general use, and yet the results shown in the table obtained from them differ in some cases nearly four hundred per cent., as for instance for the Gordon formula applied to hollow cylinders.

Results by the Crehore formula run high in value, one-half of all in the table going above those by the first three named formulas. It appears to give better results for hollow than for solid sections. But the large denominator gives small column load so that the formula errs on the safe side.

Results by the new formula are seen in every case but two, to fall between those of the first three "old reliable" formulas, and those two are outside only from one to three per cent. In comparing the means, we see that the new formula falls between the first three. If we reject the Gordon formula, which is now

going into disuse in the most approved practice, and compare the new formula with the 2d and 3d columns of results, we find the mean by the new formula falls between the means of the 2d and 3d columns. Indeed it differs but little over two per cent. from the means of the two columns. This appears the more favorable from the fact that the results in the 2d and 3d columns are obtained from formulas which are probably the best in use to-day in the engineering world.

Comparisons have not been made for other materials than wrought iron, for the reason that wrought iron is the chief material in use at present in large trusses where struts are required.

FORMULAS COMPARED WITH EXPERIMENT.

To make the comparison of the new formula with others in such a way as will be yet more satisfactory to practical men, the breaking load of 33 columns of wrought iron, boxed and open built, such as used in bridges, has been computed and placed opposite the breaking load

found by actual test in a testing machine. Also in the table is placed the calculated breaking load as found from the Rankine formula, the Bouscaren-Keystone formula, and of the Rankine formula as worked over by Prof. Baldwin for new coefficients, and given in his article above referred to.

In all cases except in Baldwin's formula, the coefficient in the numerator corresponding to the resistance to crushing, was taken at 40,000 lbs. In Baldwin's formula all the coefficients were used as given in the Magazine. The value of ε for the new formula was used, except in a few cases, as given by the experiments, when given. Corrections were made for pin friction, and for excessive stay spacing for two cases. But these probably effect the final result only 2 or 3 per cent., because the number of corrections needed were so few. But the per cent. for the individual cases was sometimes very considerable. The breaking loads stated are per square inch of cross section of column.

The experimental results cited are from a paper by G. Bouscaren to the Am. Soc. Civ. Engrs. on the strength of wrought iron columns. They are stated by some writers to be from Lovet's report, though I find no mention to this effect in the paper to the society.

The columns cited are of five distinct forms of built columns, viz.:

P. "Phoenix," hollow cylinder, four flanged segments riveted.

K. "Keystone," octagonal tube, four flanged segments riveted diametrically.

A. "American," two flanged bars riveted to the flanges of a central I beam.

S. "Square," hollow, two plates and two channel bars riveted with flanges outward.

O. "Open," two channel bars, flanges outward, latticed with slats, riveted trellis-like.

The signs in the last four columns are *prop*er to make the quantities there ap-

pearing, correct the computed values of the resistances to agree with the resistances by experiment.

The sums, + and −, appearing at the foot of the first two columns qualified with signs make it appear that the formulas give results which are too large. Hence it appears that the 40,000 used in the numerator of the Rankine and Bouscaren-Keystone formula is too high. It was used in the Rankine formula for convenience, and because it is found in the Keystone formula. But as Rankine gave the formulas the quantity was 36,000, the same as in the Baldwin formula for flat ends, for which also the denominators agree. Hence the quantities by the Baldwin formula down to No. 5 may be regarded as coinciding with Rankine's. The values in Rankine's space from No. 5 down for round ends should be reduced 10 per cent. to change from 40,000 to 36,000. In order to present the results as would be thus given for Rankine's formulas unmodified, or for the numerator at 36,000, they have been

worked out and given in the second set at the foot of Rankine's column of differences. The reduction from 40,000 to 36,000 is evidently too great as in one set the — values predominate, while in the other it is with the + values. An estimate made on this score would indicate that 38,000 would be about the best value for the Rankine formulas, the same indeed as is known to be in use by some Ry. Cos. It appears also that 40,000 is too great for the Keystone formula.

In glancing over the four last columns it is seen that in the first the — signs are chiefly among the results for flat ends, while the + signs predominate at the round ends. This indicates that the Rankine formulas are constitutionally at fault, or perhaps that the 1, $\frac{1}{9}^{6}$, and 4, coefficients to the last term of the denominator for flat, flat and round, and round, ends differ too much. As adopted in the Bouscaren-Keystone formula, viz.: 1, $\frac{2}{3}$ and 2, the discrepancies would probably be less, as indeed the column footings show. With these coefficients it would appear

that a better numerator would be about 38,000.

The column of differences due to the Baldwin formula are found to be far more satisfactory than either of the two preceding it. That is to say, the signs are more generally intermixed, the + and — values sum up more nearly equal to each other numerically, and the average discrepancy, without sign is smaller.

If, however, Nos. 20 and 34 were cut out from the table, the + and — sums would more nearly agree in value for the Rankine formula than for any of the others. But the mean error without sign would be higher than for any of the others, as indicated by the figures given Though the Baldwin formula still excels the Rankine and Keystone formulas in the smallness of mean error, yet the difference in + and — sums is greater.

But in every respect we find the last column of figures, which are due to the new formula, by far the most favorable. That is to say, the individual discrepancies in that column run lower in value

the number of + and − values are more nearly equal; the absolute values of the sums of the + and − quantities are lower and differ less; and the mean error, or discrepancy, is lowest. This, together with the fact that the theoretical values, 1, $\frac{16}{9}$ and 4 are employed, whereas in other formulas, they seem to differ too much, would indicate that the new formula is more nearly theoretically perfect than any of the others. But, in spite of these advantages, the new formula is, unfortunately, less convenient in application than the others, and the question will doubtless arise in practice, whether or no, for an important structure, to sacrifice the additional hour required for the more trustworthy results by the new formula.

As the present object is formulas for bridges, and as wrought iron is the material chiefly employed, no attempt has been made to compare the new formula with others for other materials than wrought iron. It seems probable, however, that where the coefficient of elasticity, ε, is practically constant nearly to

the point of rupture, the formula may be used without hesitation. The point of rupture is here meant to be either for rupture by tension or compression. For such a material as cast iron it may be necessary to be guided partly by empiricism, especially for long columns, where the rupture may take place on the convex side by tension, because, in tension, the ε rapidly falls off in value, as rupture is approached. But in compression, ε is more nearly constant, and hence in short columns the formula will probably apply with greater exactitude.

The great difference in the ultimate resistances to tension and compression in cast iron will also cause an approach to rupture on the convex side, relatively, much sooner than for other materials, so that the rapidly diminishing value of ε will effect much shorter columns than otherwise. For cast iron in particular, therefore, it may be necessary to use a low value of the modulus of compression and of ε, and more so as the length increases, say 60,000 to 80,000 for the

former, and 12 to 20 million for the latter.

A rational formula must certainly prove much more satisfactory for a comparatively new material, such as some of the grades of steel, than an entirely empirical formula which has not been adapted, that is, in cases where reliable values of the compressive resistance and of ε are known, and not of the constants in the empirical formulas.

MULTIPLYING EMPIRICAL FORMULAS.

Where numerous results of experiments exist for particular classes of columns, it may be possible to adopt an empirical formula for each class, or part of a class, which shall give the results for such columns, or at least to reproduce the particular values to which those formulas are adapted, with closer agreement than any single or perfectly general formula would do. But this process leads to some degree of confusion in the considerable number of formulas produced, and new experiments may develop need of additional

formulas, or require a modification of former ones.

In the above tables, comparisons were made only with formulas which are proposed to be perfectly general. But *Trautwine's Pocketbook* cites six different formulas by D. J. Whittemore, C. E., which, accordingly, would be required in computing strains for three of the forms of columns in the above table,and including only 17 of the experimental tests. Though Whittemore gives 12 different formulas, only 6 apply, and half of the experimental columns cited in the table are left by him without a formula.

In a recent paper by Prof. Wm. H. Burr, ten different formulas are produced from twenty-nine experiments. The experiments are mostly cited in the above table, and the formulas, when the whole ten are used, each for its intended form of column, give results agreeing remarkably well with the experiments. The mean error, regardless of sign, is 1720—a quantity less by about 33 per cent. than the mean by the new formula from

all the experimental columns cited in the above table, viz., 2661 (two being left out). But when we observe that in this comparison, the new formula, obtained independently of all experiment or specified material, is arraigned before 10 formulas which were obtained from the very experiments compared with, and at the rate of one formula to three experiments, it does not seem strange that the ten formulas should reduce the mean error. Considering the idiosyncracies of column tests, are we at all confident that in extending the application of the 10 formulas to 30 new tests, the mean error will remain as low as indeed either of the values cited?

FACTOR OF SAFETY IN COLUMNS.

The usual way of providing for safe loads for columns is to take a certain fractional part of the breaking load; say such a fractional part as is taken in eye bars. The fraction $\frac{1}{5}$ is common. Thus, the ultimate resistance of wrought iron

to tension is about 50,000 lbs. per square inch, and the $\frac{1}{5}$ part, or 10,000 lbs., is almost an absolute conventionalism among bridge builders, and may be regarded nearly in the sense of an absolute modulus of working resistance to tension.

Accordingly, $\frac{1}{5}$ is prefixed to column formulas (Rankine's and Gordon's) by many bridge engineers, as giving the proper fractional part of the ultimate resistance, which is to serve as the safe working resistance. But in empirical formulas, where the composition of the constants is unknown, this is a blind practice, because, from analogy with eye-bar strains, $\frac{1}{5}$ of the ultimate compressive resistance of the *material* is evidently meant, instead of $\frac{1}{5}$ of the resistance of the *column*.

This matter can be corrected in the Rational formulas, and it will be seen that the "$\frac{1}{5}$" coefficient to the empirical formulas for columns gives an unintentionally small safe load. A glance at formulas (114) and (120) will suffice for this. Thus, (114) shows that by using $\frac{1}{5}$

of the ultimate resistance to compression, t, we effect both the numerator and the second term of the denominator, whereas, in (120), we only modify the numerator by taking $\frac{1}{5}$ of the breaking load by formula.

To indicate how the "safe load" is effected by the empirical as compared with the rational method, the following values have been computed for certain ones of the columns in the principal table above.

	No.	l	$\frac{l}{k}$	For $\frac{1}{5}$ Column breaking load.	For $\frac{1}{5}$ t in (114)
Flat end	6	15	5.2	7080	7790
"	23	24	7.1	6540	7680
"	29	28	9.3	5860	7870
Rounded	16	20	8.5	4800	6320
"	36	34	12.	3960	5325

Here the last column of figures is to be regarded as truly representing the resistance of the columns, per square inch of section, when the maximum strain on the material of the column is only the fifth part of the breaking strain, while the column of figures preceding the last, usually regarded as the fifth part of the

breaking load, does not strain the material to more than about one-seventh of its ultimate resistance. But fortunately this difference is on the safe side, so that the majority of existing columns in bridges are amply strong.

The figures in the last column are all less than 8000 lbs., the maximum strain assumed as allowable, that is, the fifth part of 40,000 lbs. The differences are to be considered as provisional, for preventing the strains rising in excess of the 8000 lbs. by buckling or otherwise.

By taking $\frac{1}{7}t$ in place of $\frac{1}{8}t$ for computing the last column of figures, perhaps as close an agreement is obtained with the column preceding it as is possible with a simple fraction. From this fact it appears that the "factor of safety" usually employed for columns has been in the neighborhood of $\frac{1}{7}$, or that the working resistance of iron in columns has been from 5000 to 6000 lbs. per square inch, compression. But as a high value of the working resistance to com-

pression is regarded by some prominent engineers as safer than a like high value for tension, it appears that columns have habitually been made unduly heavy.

SEMI-COLUMNS, WITH PIN ENDS AND COMPRESSION.

Perhaps this term will serve to distinguish pieces which are subject to longitudinal and transverse strains; as in Fig. 2, or in Fig. 28 and following.

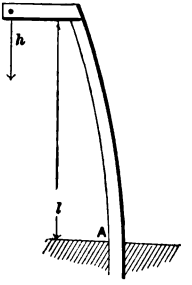


Figure 28.

An expression for the max. strain in the section at A, Fig. 28, is obtained by *combining* Eq. (8) with (25) and *eliminating* M. Whence

$$t=\frac{\mathrm{T}hd_1}{\mathrm{I}\cos nl}\pm\frac{\mathrm{T}}{\mathrm{K}} \quad . \quad . \quad . \quad (124).$$

$$\left.\begin{array}{l}\text{where } n^2=\dfrac{\mathrm{T}}{\varepsilon\mathrm{I}}=\dfrac{t'\mathrm{K}}{\varepsilon\mathrm{I}}=\dfrac{t}{\varepsilon k^2}\\ \text{also } \dfrac{\mathrm{T}}{\mathrm{K}}=t', \text{and } \dfrac{\mathrm{T}}{\mathrm{I}}=\dfrac{t'}{k^2}\end{array}\right\} \quad . \quad (124a).$$

These quantities may, indeed, be introduced with probable greater convenience for application, giving

$$\frac{t}{t'}=\frac{hd_1}{k^2\cos\frac{l}{k}\sqrt{\frac{t'}{\varepsilon}}}\pm 1 \quad . \quad . \quad (125).$$

the + sign being used to obtain the max. compressive strain, and the — sign for the max. tensile strain. The piece will fail from the first or second, as depending on the nearest approach of that strain to the modulus of crushing or of rupture. For wrought iron the + sign should be used, while for cast iron, except when h and l are very short, the — sign. The max. strain per square inch is t, tension or compression as just indicated, while t' is the actual average compressive force

which is applied to the semi-column per square inch of the cross section. For instance, if the section be 10 square inches and the compressive force applied be 100,000 lbs., $t=$ 10,000 lbs. But the max. t' resulting strain in the semi-column may be $t=$20,000 lbs. per square inch. Also h is the distance from the center of gravity of the end cross section to where the compressive load is applied; d_1 the distance from the center of gravity of the middle section to the side, concave or convex, where the strain t is reckoned; k the radius of gyration of cross section; l the length of semi column, and ε the coefficient of elasticity.

When d_1 is taken on the convex side, and the 1st term 2d member $= 1$, we will have tension $t=o$.

Though a direct determination of t is effected for an assumed value of t', yet the converse will doubtless most frequently be desired, but cannot be obtained by a direct simple solution. A few trial values of t' however will give a sufficiently close value for t.

For a case like Fig. 29 it is only necessary to put $\frac{l}{2}$ instead of l in (125); whence, for AB=l, AC=h, and for pin bearings at C and D,

Figure 29.

$$\frac{t}{t'}=\frac{hd_1}{k^2\cos\frac{l}{2k^2}\sqrt{\frac{t'}{\varepsilon}}}\pm 1 \quad . \quad (126).$$

If there be a load, P, at the middle of the semi-column, acting transversely and downward in Fig. 29, a suitable expression is obtained by substituting M from (23) in (8), observing that for (23), l and P should be changed to $\frac{l}{2}$ and $\frac{P}{2}$ and that the essential sign of M is —.

Hence, for pin bearings, and a load P at the middle

$$t=\frac{d_1}{I}\left(\frac{P}{2n}\text{tang } n\frac{l}{2} \quad \frac{Th}{\cos n\frac{l}{2}}\right)\pm\frac{T}{K} \quad . . \quad (127).$$

in which I = the principal moment of inertia of the cross section K, T the whole compressive force applied, and n as given above. P is — for acting upward.

In order that the flexural moment at the middle of the semi-column be zero, the parenthesis should be zero. Whence

$$h=-\frac{P}{2nT}\sin n\frac{l}{2} \quad . \quad . \quad . \quad (128).$$

This value of h is expected to be negative, and it implies that when P acts downward in Fig. 29, the arms h=BD= AC must be laid off downward.

This equation is applicable to upper chords of deck bridges which carry a floorbeam at the middle of each panel, and for determining the position of the pin bearing such that the max. load shall only cause compressive strain in the middle of the panel length. The chord *should* however be considerably heavier *than as thus* determined.

By making $h=o$ in (127), we obtain the expression for a column with pin bearings against the middle of which there exists the lateral thrust P, viz.:

$$=\frac{Pd_1}{2nI}\text{tang}\, d\frac{l}{2}\pm\frac{T}{K} \quad . \quad . \quad . \quad (129).$$

In deck bridges where numerous floor beams rest on each panel length of the upper chord, as is sometimes the case with wooden floor beams on iron trusses, we have each panel length serving like a column under cross loading, the latter being nearly uniformly distributed.

An expression for such a case is obtained from (35) and (8). Hence for pin bearings

$$t=\frac{wd_1}{n^2I}\text{tang}\, n\frac{l}{4}\text{tang}\, n\frac{l}{2}\pm\frac{T}{K} \quad . \quad . \quad (130).$$

$$=\frac{wd_1}{n^2I}\left(\sec\ n\frac{l}{2}-1\right)\pm\frac{T}{K}$$

where w is the loading per unit length.

With Fixed Ends and Compression.—Where the semi-column is fixed at the ends, as in continuous upper chords,

combine (77) with (8) for a load P at the middle ; whence

$$t=\frac{\mathrm{P}d_1}{2n\mathrm{I}}\mathrm{tang}\,n\frac{l}{4}\pm\frac{\mathrm{T}}{\mathrm{K}} \quad . \quad . \quad . \quad (131).$$

which gives the strain at either the middle or end of the panel length.

For a uniformly distributed load, w, for unit length, and for continuous upper chords, we have by (78) and (8)

$$t=\frac{wd_1}{n^2\mathrm{I}}\left(1-n\frac{l}{2}\cot n\frac{l}{2}\right)\pm\frac{\mathrm{T}}{\mathrm{K}} \quad . \quad . \quad . \quad . \quad (132).$$

for the end of the panel length, and where the strain t is greatest.

At the middle we will have from $x=\frac{l}{2}$ in (75) and (8)

$$t=\frac{wd_1}{n^2\mathrm{I}}\left\{\frac{n\frac{l}{2}}{\sin n\frac{l}{2}}-1\right\}\pm\frac{\mathrm{T}}{\mathrm{K}} \quad . \quad . \quad . \quad (133).$$

These formulas for distributed loads apply to account for the strain due to the weight of the parts, themselves, as in up-

per chords of through bridges; and the inclined end posts. In the latter however gravity acts with a transverse component only, which is considerably less than where the piece is horizontal.

Other cases, such as for one pin bearing and one fixed end, is provided for in (63), (64), (65) and (8), two floor beams and pin bearings in (49), &c., &c.

Pin Bearings and Tension.—In lower

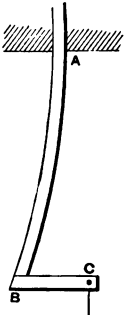


Figure 30.

where the longitudinal force is tension, chords of bridges and other cases

we may combine (20) with (8), and obtain an expression adapted to Fig. 30. viz.:

$$t=\frac{d_1}{I}\cdot\frac{2Th}{e^{nl}+e^{-nl}}\pm\frac{T}{K} \quad . \quad . \quad . \quad . \quad (134),$$

where l=AB, and h=BC. Here d_1 will usually be measured on the convex side of beam, and where the + sign will generally be required. Also t will probably always be tension.

The formula for Fig. 31, will be ob-

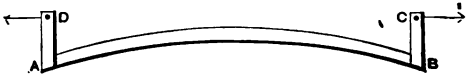


Figure 31.

tained by replacing l in (134) by $\frac{l}{2}$, giving for pin bearings at C and D,

$$t=\frac{d_1}{I}\cdot\frac{2Th}{e^{n\frac{l}{2}}+e^{-n\frac{l}{2}}}\pm\frac{T}{K} \quad . \quad . \quad . \quad . \quad (135).$$

in which l=AB, and h=AD.=BC.

When a load is applied at the middle,

acting downward, we have, by aid of (19) and (8), observing that P in (19) is the half of P here,

$$t=\frac{\mathrm{P}d_1}{2n\mathrm{I}}\frac{\left(e^{n\frac{l}{2}}-e^{-n\frac{l}{2}}\right)-2\mathrm{T}h}{e^{n\frac{l}{2}}+e^{-n\frac{l}{2}}}\pm\frac{\mathrm{T}}{\mathrm{K}} \quad . \quad . \quad (136).$$

In order that the flexural moment at the middle of the beam shall be zero, the 1st term, 2d member, should be zero, which condition determines h.

Also h may be zero for a link or eye bar.

For a uniformly distributed load of w per unit length, we have for pin bearings

$$t=\frac{wd_1}{n^2\mathrm{I}}\left\{1-\frac{2}{e^{n\frac{l}{2}}+e^{-n\frac{l}{2}}}\right\}\pm\frac{\mathrm{T}}{\mathrm{K}} \quad . \quad . \quad (137.$$

Fixed Ends and Tension.—For fixed ends we may combine (72) and (8) for a load at the middle, and for a distributed load (73); and these give strains at the end of the panel length. For the middle employ (74).

Other cases may readily be provided for. For instance, for two floor beams to the panel length, the same resting on the lower chord, use (43) for ends in pin bearings.

For one pin bearing and one fixed end we have (68) and (69), &c., &c. The value of M found from any of these equations may be introduced in eq. (8) to the end of obtaining the maximum strain t, either tension or compression. Equivalents expressed in (124a) are applicable in all.

Examples of Strains in Semi-Columns.—The following examples have been worked to indicate the use of the formulas.

Take a wrought iron end post, or a panel length of an upper chord, length 20 feet, constructed as shown in Fig. 32, of two 10″ channel bars of $\frac{1}{2}'' \times 2\frac{1}{2}''$ flanges $\times \frac{8}{8}''$ web; a plate $12'' \times \frac{8}{8}''$, and slats, all riveted together as shown. At the ends, proper thickening pieces, *gussets, &c.*, are supposed in use.

Results for the section, with moment

of inertia as dotted, are exclusive of latticing,

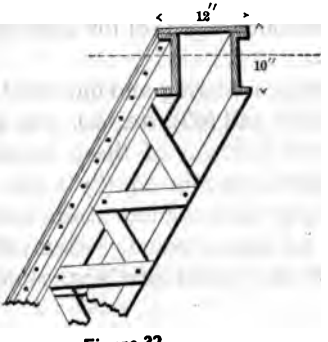


Figure 32.

K=section=12.875 sq. in.
I=205, axis 3″ from plate.
k^2=15.93 in.=radius of gyration.
d_1=3.36 in. plate side.
" =7. " opposite side.
w=wt. per inch length 3.6 lbs.

As a vertical column with knife bearings or pins without friction at center of gravity of sec. and with a max. strain *of 10,000 lbs.* per square inch, the resist-

ance by new formula = 9011 lbs. for each inch of section.

As a horizontal column with latticing downward, knife bearings, &c., its weight, causes an additional strain of 459 lbs. per square inch, as per first member of eq. (130). This is to be added to $\frac{T}{K}$ = either the 10,000 or 9,011 lbs., according to whether the former or latter is to be regarded as the strain per square inch without flexure. The difference between these figures is the provisional excess mentioned under "factors of safety," and is peculiar to columns. This excess should grow rapidly less as we change from partially indifferent column-like strains to positive moments, such as provided for in eqs. (124) and (126).

The strain computed from the ordinary formula for a beam supported at its ends, its weight being the load, and T= *o*, is 404. lbs., a quantity less than the above, as it should be.

We observe that the strain due to the *uniform* load *w* is, by eq. (130) directly

proportional to w when T is constant, so that if this load be increased to 40 lbs. per inch or 480 lbs. per ft., the strain will be increased to 5,559 lbs. per sq. in., while by the ordinary formula, ignoring T, it is 4,894 lbs. As an end post inclined at 60° with the horizon the strains will be reduced by a half.

Next, let the loading be P at the middle, of 1,000 lbs., with T and other data as before. As a column in pin bearings, Eq. (127), with $h=o$, gives for the strain due to P 1,090. lbs. per sq. in. By the ordinary formula it is 985.

Transferring the pins to the top of end sections, other conditions as in the last, then $h=3.36$ in. in (127), and the strain due to 1st term of 2d member of (127) is $1090+578=1668$.

To find where to place the pins so that the moment of strain and the middle shall be zero, we have from (128) $h=-6.34$ inches, which is within $\frac{2}{3}$ of an inch of the bottom.

As a last example, let the aggregate *section of* an iron lower chord be 4'

breadth by 5 inches depth. Let the ends be fixed, as when continuous and clamped at the panel points. Let the panel length carry two floor beams, each with center 18″ from panel point. Suppose the severest strain be due to the drive wheels, and when one pair rests directly on one of the floor beams. Then the other wheels will probably not rest on the other floor beam for unequal spacing. To simplify the work, find the effective load on the floor beam directly under the pair of drive wheels, and let it amount to 23,560 lbs. = Q. Also take the point of contrary flexure near the other floor beam as a point of support. This places the chord in the condition of a beam fixed at one end, supported at the other, and having a load applied at 18″ from the fixed end. Take the beam thus conditioned at l = 129.6″ and let T = 88,000 lbs., ε = 24,000,000. Then,

$$\frac{T}{\varepsilon I} = .000088 = n^2 \qquad n = .009381$$

$$nl = 1.2158 \qquad na = 0.16886$$

$n(l-a)=1.0469$

$e^{nl}=3.3729$ $\qquad$ $e^{-nl}=.2965$

$e^{na}=1.1839$ $\qquad$ $e^{-na}=.8446$

$e^{n(l-a)}=2.8489$ $\qquad$ $e^{-n(l-a)}=.3510$

Then by (69)

$$\frac{P}{Q}=.02968$$

The maximum strain is at the fixed end, where x and $y=o$. Hence, the first member of 68, by aid of P from above, gives

$$M=Qa-Pl=14.153\ Q=333500.$$

$$\text{also}\frac{T}{K}=\frac{88000}{20}=4400,$$

These in eq. 8 give the max. strain per square inch desired,

$$t=\frac{2.5}{41.67}333500+4400=24405.\ \text{lbs.}$$

The ordinary formulas for the same case and data, computing the parts separately and adding give

$$t=24800.\ \text{lbs.}$$

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